129 Lecture Notes
Relativistic Quantum Mechanics

1 Need for Relativistic Quantum Mechanics

The interaction of matter and radiation field based on the Hamitonian

$$H = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} - \frac{Ze^2}{r} + \int d\vec{x} \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2). \tag{1}$$

(Coulomb potential is there only if there is another static charged particle.) The Hamiltonian of the radiation field is Lorentz-covariant. In fact, the Lorentz covariance of the Maxwell equations is what led Einstein to propose his special theory of relativity. The problem here is that the matter Hamitonian which describes the time evolution of the matter wave function is not covariant. A natural question is: can we find a new matter Hamiltonian consistent with relativity?

The answer turned out to be yes and no. In the end, a fully consistent formulation was not obtained by modifying the single-particle Schrödinger wave equation, but obtained only by going to quantum field theory. We briefly review the failed attempts to promote Schrödinger equation to a relativistically covariant one. Despite the failure, it resulted in the prediction that anti-matter exists, which was beautifully confirmed experimentally.

2 Klein–Gordon Equation

The Schrödinger equation is based on the non-relativisitc expression of the kinetic energy

$$E = \frac{\vec{p}^2}{2m}. \tag{2}$$

By the standard replacement

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \vec{p} \to -i\hbar\vec{\nabla}, \tag{3}$$

we obtain the Schrödinger equation for a free particle

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\Delta}{2m}\psi. \tag{4}$$

A natural attempt is to use the relativistic version of Eq. (2), namely

$$\left(\frac{E}{c}\right)^2 = \vec{p}^2 + m^2c^2. \tag{5}$$

Then using the same replacements Eq. (3), we obtain a wave equation

$$\left(\frac{\hbar}{c}\frac{\partial}{\partial t}\right)^2 \phi = (\hbar^2\Delta - m^2c^2)\phi. \tag{6}$$

It is often written as

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0, \tag{7}$$

where $\Box = \partial_\mu\partial^\mu = (\frac{1}{c}\partial_t)^2 - \Delta$ is called D'Alambertian and is Lorentz-invariant. This equation is called Klein–Gordon equation.

You can find plane-wave solutions to the Klein–Gordon equation easily. Taking $\phi = e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}$, Eq. (6) reduces to Eq. (5). Therefore, as long as energy and momentum follows the Einstein's relation Eq. (5), the plane wave is a solution to the Klein–Gordon equation. So far so good!

The problem arises when you try to rely on the standard probability interpretation of Schrödinger wave function. If a wave function ψ satisfies Schrödinger equation Eq. (4), the total probability is normalized to unity

$$\int d\vec{x}\psi^*(\vec{x},t)\psi(\vec{x},t) = 1. \tag{8}$$

Because the probability has to be conserved (unless you are interested in seeing 5 times more particles scattered than what you have put in), this normalization must be independent of time. In other words,

$$\frac{d}{dt}\int d\vec{x}\psi^*(\vec{x},t)\psi(\vec{x},t) = 0. \tag{9}$$

It is easy to see that Schrödinger equation Eq. (4) makes this requirement satisfied automatically thanks to Hermiticity of the Hamiltonian.

On ther other hand, the probability defined the same way is not conserved for Klein–Gordon equation. The point is that the Klein–Gordon equation is second order in time derivative, similarly to the Newton's equation of motion in mechanics. The initial condtions to solve the Newton's equation of motion are the initial positions and initial velocities. Similarly, you have to give both initial configuration $\phi(\vec{x})$ and its time derivative $\dot{\phi}(\vec{x})$ as the initial conditions at time t. The time derivative of the "total probability" is

$$\frac{d}{dt}\int d\vec{x}\phi^*(\vec{x},t)\phi(\vec{x},t) = \int d\vec{x}(\dot{\phi}^*(\vec{x},t)\phi(\vec{x},t) + \phi^*(\vec{x},t)\dot{\phi}(\vec{x},t)), \tag{10}$$

and ϕ and $\dot{\phi}$ are independent initial conditions, it in general does not vanish, and hence the "total probability" is not conserved. In other words, this is an unacceptable definition for the probability, and the standard probability interpretation does not work with Klein–Gordon equation.

One may then ask, if there is a conserved quantity we can possibly call "probability." It is easy to see that the following quantity is conserved:

$$\int d\vec{x}(i\phi^*\dot{\phi} - i\dot{\phi}^*\phi) \tag{11}$$

using the Klein–Gordon equation. However, this quantity cannot be called probability either because it is not positive definite.

Overall, the Klein–Gordon equation appears to be a good relativistic replacement for the non-relativistic Schrödinger equation at the first sight, but it completely fails to give the conventional probability interpretation of a single-particle wave function. In other words, the Klein–Gordon equation, if useful at all, does not describe the probability wave, which the Schrödinger equation does, but describes something else. Because of this reason, the Klein–Gordon equation was abandoned for a while. We will come back to the question what it actually describes later on.

3 Dirac Equation

3.1 Heuristic Derivation

Dirac was the first to realize the problem with the probability interpretation for equations with second-order time derivatives. He insisted on finding an equation with only first-order time derivatives. Because the relativity requires to treat time and space on equal footing, it means that the equation has to be only first-order in spatial derivatives, too. Given the replacements Eq. (3), the Hamiltonian must be linear in the momentum. Then the only equation you can write down is of this form:

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi = [c\vec{\alpha}\cdot\vec{p} + mc^2\beta]\psi. \tag{12}$$

At this point, we don't know what $\vec{\alpha}$ and β are. The Dirac further required that this equation gives Einstein's dispersion relation $E^2 = \vec{p}^2c^2 + m^2c^4$ like the Klein–Gordon equation. Because the energy E is the eigenvalue of the Hamitonian, we act H again on the Dirac wave function and find

$$H^2\psi = [c^2\alpha^i\alpha^j p^i p^j + mc^3(\alpha^i\beta + \beta\alpha^i)p^i + m^2c^4\beta^2]\psi. \tag{13}$$

In order for the r.h.s. to give just $\vec{p}^2c^2 + m^2c^4$, we need

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}, \qquad \beta^2 = 1, \qquad \alpha^i\beta + \beta\alpha^i = 0. \tag{14}$$

These equations can be satisfied if α^i, β are *matrices*! Setting the issue aside why the hell we have to have matrices in the wave equation, let us find solutions to the above equations. There are of course infinite number of solutions related by unitary rotations, but the canonical choice Dirac made was

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

They are four-by-four matrices, and σ^i are the conventional Pauli matrices. You can easily check the relations Eq. (14) using the matrices in Eq. (15). Correspondingly, the wave function ψ must be a four-component column vector. We will come back to the meaning of the multi-component-ness later. But the first point to check is that this equation does allow a conserved probability

$$i\hbar\frac{d}{dt}\int d\vec{x}\psi^\dagger\psi = \int d\vec{x}[\psi^\dagger(H\psi) - (H\psi)^\dagger\psi] = 0, \tag{16}$$

simply because of the hermiticity of the Hamiltonian (note that $\vec{\alpha}$, β matrices are hermitean). This way, Dirac found a wave equation which satisfies the relativistic dispersion relation $E^2 = \vec{p}^2c^2 + m^2c^4$ while admitting the probability interpretation of the wave function.

3.2 Solutions to the Dirac Equation

Let us solve the Dirac equation Eq. (12) together with the matrices Eq. (15). For a plane-wave solution $\psi = u(p)e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}$, the equation becomes

$$\begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix} u(p) = Eu(p). \tag{17}$$

This matrix equation is fairly easy to solve. The first point to note is that the matrix $\vec{\sigma}\cdot\vec{p}$ has eigenvalues $\pm|\vec{p}|$ because $(\vec{\sigma}\cdot\vec{p})^2 = \sigma^i\sigma^j p^i p^j = \frac{1}{2}\{\sigma^i,\sigma^j\}p^i p^j = \delta^{ij}p^i p^j = \vec{p}^2$. Using polar coordinates $\vec{p} = |\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we find

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = |\vec{p}| \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}, \tag{18}$$

and their eigenvectors

$$\vec{\sigma}\cdot\vec{p}\chi_+(\vec{p}) = \vec{\sigma}\cdot\vec{p}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = +|\vec{p}|\chi_+(\vec{p}), \tag{19}$$

$$\vec{\sigma}\cdot\vec{p}\chi_-(\vec{p}) = \vec{\sigma}\cdot\vec{p}\begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} = -|\vec{p}|\chi_-(\vec{p}). \tag{20}$$

Once $\vec{\sigma}\cdot\vec{p}$ is replaced by eigenvalues $\pm|\vec{p}|$, the rest of the job is to diagonalize the matrix

$$\begin{pmatrix} mc^2 & \pm|\vec{p}|c \\ \pm|\vec{p}|c & -mc^2 \end{pmatrix}. \tag{21}$$

This is easily done using the fact that $E = \sqrt{|\vec{p}|^2c^2 + m^2c^4}$. In the end we find two eingenvectors

$$u_+(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\chi_+(\vec{p}) \\ \sqrt{\frac{E-mc^2}{2mc^2}}\chi_+(\vec{p}) \end{pmatrix}, \qquad u_-(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\chi_-(\vec{p}) \\ -\sqrt{\frac{E-mc^2}{2mc^2}}\chi_-(\vec{p}) \end{pmatrix}. \tag{22}$$

In the non-relativistic limit $E \to mc^2$, the upper two components remain $O(1)$ while the lower two components vanish. Because of this reason, the upper two components are called "large components" while the lower two "small components." This point will play an important role when we systematically expand from the non-relativistic limit.

An amazing thing is that there are two solutions with the same momentum and energy, and they seem to correspond to two spin states. Then the wave equation describes a particle of spin 1/2! In order to make this point clearer, we look at the conservation of angular momentum. The commutator

$$[H, L^i] = [c\vec{\alpha}\cdot\vec{p} + mc^2\beta, \epsilon_{ijk}x^j p^k] = -i\hbar c\epsilon_{ijk}\alpha^j p^k \neq 0 \tag{23}$$

does not vanish, and hence the orbital angular momentum is not conserved. On the other hand, the matrix

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{24}$$

has the commutator

$$[H, \Sigma^i] = [c\vec{\alpha} \cdot \vec{p} + mc^2\beta, \Sigma^i] = cp^j[\alpha^j, \Sigma^i] = -2i\epsilon_{ijk}cp^j\alpha^k. \tag{25}$$

Therefore, the sum

$$\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma} \tag{26}$$

commutes with the Hamiltonian and hence is conserved. Clearly, the matrix $\frac{\hbar}{2}\vec{\Sigma}$ has eigenvalues $\pm\frac{\hbar}{2}$ and hence corresponds to spin 1/2 particle. The eigenvectors $u_\pm(p)$ we obtained above are also eigenvectors of $\vec{\Sigma} \cdot \vec{p} = \pm|\vec{p}|$ by construction, and hence $\vec{J} \cdot \vec{p} = \frac{\hbar}{2}\vec{\Sigma} \cdot \vec{p} = \pm\frac{\hbar}{2}|\vec{p}|$. In other words, they are *helicity* eigenstates $(\vec{J} \cdot \vec{p})/|\vec{p}| = \pm\frac{\hbar}{2}$. Helicity is the angular momentum projected along the direction of the momentum, where the orbital angular momentum trivially drops out because of the projection. And hence the helicity is purely spin. This analysis demonstrates that the Dirac equation indeed describes a particle of spin 1/2 as guessed above.

This line of reasoning is fascinating. It is as if the conservation of probability requires spin 1/2. Maybe that is why all matter particles (quarks, leptons) we see in Nature have spin 1/2!

But the equation starts showing a problem here. The Dirac wave function ψ has four components, while we have obtained so far only two solutions. There must be two more independent vectors orthogonal to the ones obtained above. What are they? It turns out, they correspond to *negative energy* solutions. Writing $\psi = v(p)e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar}$, the vectors $v(p)$ must satisfy the following matrix equation similar to Eq. (17) but with the opposite sign for the mass term

$$\begin{pmatrix} -mc^2 & c\vec{\sigma} \cdot \vec{p} \\ c\vec{\sigma} \cdot \vec{p} & mc^2 \end{pmatrix} v(p) = Ev(p). \tag{27}$$

Therefore the solutions are obtained in the same manner but the upper two and lower two components interchanged

$$v_+(p) = \begin{pmatrix} \sqrt{\frac{E-mc^2}{2mc^2}}\chi_+(\vec{p}) \\ \sqrt{\frac{E+mc^2}{2mc^2}}\chi_+(\vec{p}) \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} -\sqrt{\frac{E-mc^2}{2mc^2}}\chi_-(\vec{p}) \\ \sqrt{\frac{E+mc^2}{2mc^2}}\chi_-(\vec{p}) \end{pmatrix}. \tag{28}$$

Note that the definition $\psi = v(p)e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar}$ has the energy and momentum in the plane wave with the opposite sign from the normal one, and hence positive $E = \sqrt{|\vec{p}|^2c^2 + m^2c^4}$ means *negative* energy solution. There is no reason to prefer positive energy solutions over negative energy ones as far as the Dirac equation itself is concerned.

What is wrong with having negative energy solutions? For example, suppose you have a hydrogen atom in the $1s$ ground state. Normally, it is *the* ground state and it is absolutely stable because there is no lower energy state it can decay into. But with the Dirac equation, the story is different. There are *infinite* number of negative energy solutions. Then the $1s$ state can emit a photon and drop into one of the negative energy states, and it happens *very* fast (it is of the same order of magnitude as the $2p$ to $1s$ transition and hence happens within 10^{-8} sec for a *single* negative energy state. If you sum over all final negative-energy states, the decay rate is infinite and hence the lifetime is zero)! Such a situation is clearly unacceptable.

3.3 Anti-Matter

Dirac is ingenious not just to invent this equation, but also to solve the problem with the negative energy states. He proposed that all the negative energy states are already filled in the "vacuum." In his reasoning, the $1s$ state cannot decay into any of the negetive energy states because they are already occupied, thanks to Pauli's exclusion principle. It indeed makes the $1s$ state again absolutely stable. Now the equation is saved again. The "vacuum" with all the negative energy states (an infinite number of them) occupied is called the "Dirac sea."

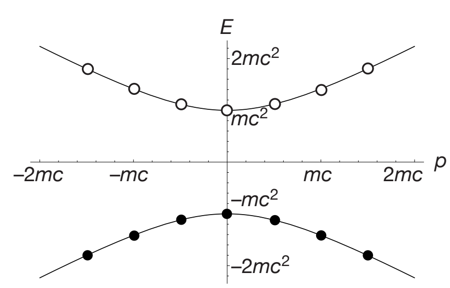

Figure 1: Dirac sea. All negative energy states are filled, while the positive energy states are not.

If you put an electron in one of the positive energy states, that is a normal electron with normal dispersion relation $E = \sqrt{p^2c^2 + m^2c^4}$. One the other hand, you can remove one electron from the Dirac sea. Let us remove an electron of momentum $\vec{p}$ and energy $-\sqrt{\vec{p}^2c^2 + m^2c^4}$. Relative to the Dirac sea, the state has momentum $-\vec{p}$ because you have removed the momentum $\vec{p}$. The energy is positive, because you have removed a negative energy $-\sqrt{\vec{p}^2c^2 + m^2c^4}$. Therefore, this state has a normal dispersion relation. An important point is that you also have removed the electric charge $e < 0$ of the electron. Therefore this state has the electric charge $-e > 0$. What it means is that this is a particle of positive charge $-e$ with momentum $-\vec{p}$ and energy $\sqrt{\vec{p}^2c^2 + m^2c^4}$. This is a new particle, "positron." It is the *anti-particle* of the electron. Dirac theory hence predicts the existence of an anti-particle for any spin 1/2 particles.[1]

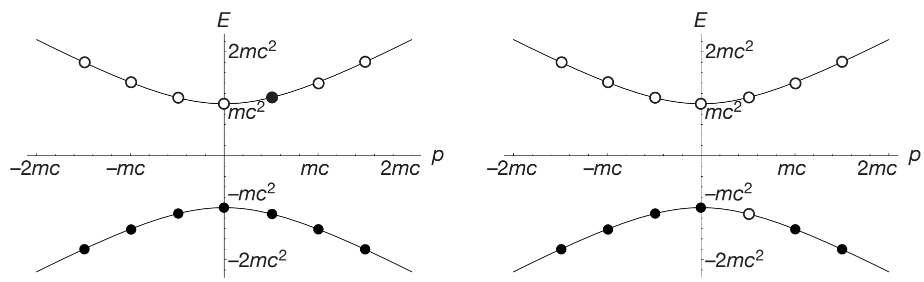

Figure 2: An electron in a positive energy state is a normal electron. If you remove an electron from one of the negative energy states, the state has a positive electric charge and is a positron.

If you inject enough energy, you may excite an electron in the negative energy state to a positive energy state. Then you have created a pair of an electron and a positron. However, the excited electron will eventually decay back down to the negative energy state by releasing energy (typically two of three photons). This is a "pair annihilation" process. Pair creation and pair annihilation are common phonemena in high-energy physics.

But there is a catch with the "Dirac sea." We wanted to find a single-particle wave function which is consistent with both relativity and probability interpretation. The Dirac equation indees seems to be consistent both with relativity and probability interpretation. But the correct implementation calls for a multi-body state (actually, an infinite-body state)! We can't just talk about a single particle wave function $\psi(\vec{x})$ for a single electron, but

[1]Dirac himself, being afraid of predicting a non-existing particle, initially claimed that this positively charged hole must be the proton. But other people, notably Robert Oppenheimer, pointed out that the hole must have the same mass as the electron.

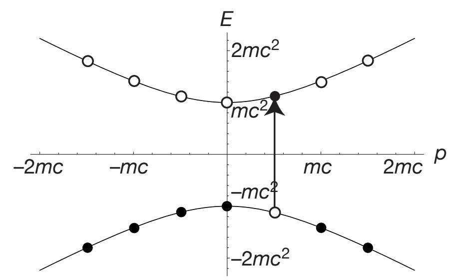

Figure 3: An electron in a negative energy state is excited to a positive energy state. This is a pair creation of an electron and a positron.

only a multi-particle one $\psi(\vec{x}; \vec{y}_1, \vec{y}_2, \cdots)$ with an inifinite number of negative energy electrons at positions $\vec{y}_k$. What it means is that we can't talk about single-particle wave mechanics in the end.

The hope for a good-old single-particle Schrödinger-like wave mechanics is gone. We couldn't do it with the Klein–Gordon equation because it didn't allow probability interpretation. We couldn't do it with the Dirac equation either because it ended up as a multi-particle problem. In the end, the only way to go is the quantum field theory. But most of discussions can be made without referring to rigorous formalism of quantum field theory. We stay away from it for the purpose of this course.

3.4 Discovery of Positron

Indeed the positron was discovered in cosmic rays by Carl D. Anderson in 1932. This was the first anti-particle. The paper is *Phys. Rev.* **43**, 491–494 (1933). You are encouraged to read the paper.

Everybody particle species in nature has their anti-particle. Proton comes with anti-proton, which was discovered in Berkeley. Quarks, constituents of protons and neutrons, also have anti-quarks counterparts. Some particles are anti-particles of their own. Photon is the anti-particle of its own. It is still a big question if neutrino is the anti-particle of its own. We don't know the answer yet.

3.5 Coupling to the Radiation Field

The interaction between the Dirac field and the Electromagnetic Field follows the same rule in the Schrödinger theory $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$, or equivalently, $-i\hbar\vec{\nabla} \to -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}$. Its Lorentz-covariant generalization also determines the time-derivative: $i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{1}{c}\frac{\partial}{\partial t} - \frac{e}{c}\phi$. (The relative sign difference is due to the fact that $A_\mu = (\phi, -\vec{A})$ transforms the same way as the derivative $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$.) Therefore, the Dirac equation is

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi - c\vec{\alpha}\cdot(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}) - mc^2\beta\right)\psi. \tag{29}$$

For stationary states, we are interested in solving the equation

$$\left[c\vec{\alpha}\cdot\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right) + mc^2\beta + e\vec{A}^0\right]\psi = E\psi. \tag{30}$$

The way we will discuss it is by a sytematic expansion in $\vec{v} = \vec{p}/m$. It is basically a non-relativisic approximation keeping only a few first orders in the expansion. Let us write Eq. (30) explicitly in the matrix form, and further write $E = mc^2 + E'$ so that E' is the energy of the electron on top of the rest energy. We obtain

$$\begin{pmatrix} e\phi & c\vec{\sigma}\cdot(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}) \\ c\vec{\sigma}\cdot(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}) & -2mc^2 + e\phi \end{pmatrix}\psi = E'\psi. \tag{31}$$

The solution lives mostly in the large components, *i.e.*. the upper two components in ψ. The equation is diagonal in the absence of $\vec{\sigma}\cdot(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A})$, and we can regard it as a perturbation and expand systematically in powers of it. To simplify notation, we will write $\vec{p} = -i\hbar\vec{\nabla}$, even though it must be understood that we are not talking about the "momentum operator" $\vec{p}$ acting on the Hilbert space, but rather a differential operator acting on the field ψ. Let us write four components in terms of two two-component vectors,

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}, \tag{32}$$

where the large component χ is a two-component vector describing a spin two particle (spin up and down states). η is the small component which vanishes in the non-relativistic limit. Writing out Eq. (31) in terms of χ and η, we obtain

$$e\phi\chi + c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})\eta = E'\chi \tag{33}$$

$$c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})\chi + (-2mc^2 + e\phi)\eta = E'\eta. \tag{34}$$

Using Eq. (34) we find

$$\eta = \frac{1}{E' + 2mc^2 - e\phi} c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})\chi. \tag{35}$$

Substituting it into Eq. (34), we obtain

$$e\phi\chi + c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})\frac{1}{E' + 2mc^2 - e\phi} c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})\chi = E'\chi. \tag{36}$$

In the non-relativistic limit, $E', e\phi \ll mc^2$, and hence we drop them in the denominator. Within this approximation (called Pauli approximation), we find

$$e\phi\chi + \frac{[\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})]^2}{2m}\chi = E'\chi. \tag{37}$$

The last step is to rewrite the numerator in a simpler form. Noting $\sigma^i\sigma^j = \delta^{ij} + i\epsilon_{ijk}\sigma^k$,

$$\begin{aligned}
[\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A})]^2 &= (\delta^{ij} + i\epsilon_{ijk}\sigma^k)(p^i - \frac{e}{c}A^i)(p^j - \frac{e}{c}A^j) \\
&= (\vec{p} - \frac{e}{c}\vec{A})^2 + \frac{i}{2}\epsilon_{ijk}\sigma^k[p^i - \frac{e}{c}A^i, p^j - \frac{e}{c}A^j] \\
&= (\vec{p} - \frac{e}{c}\vec{A})^2 + \frac{ie}{2c}\epsilon_{ijk}\sigma^k i\hbar(\nabla_i A^j - \nabla_j A^i) \\
&= (\vec{p} - \frac{e}{c}\vec{A})^2 - \frac{e\hbar}{c}\vec{\sigma}\cdot\vec{B}. \qquad (38)
\end{aligned}$$

Then Eq. (37) becomes

$$\frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m}\chi - 2\frac{e\hbar}{2mc}\vec{s}\cdot\vec{B} + e\phi\chi = E'\chi. \tag{39}$$

In other words, it is the standard non-relativistic Schrödinger equation except that the g-factor is fixed. The Dirac theory predicts $g = 2$! This is a great success of this theory.