129 Lecture Notes
More on Dirac Equation

1 Ultra-relativistic Limit

We have solved the Diraction in the Lecture Notes on Relativistic Quantum Mechanics, and saw that the upper (lower) two components are large (small) in the non-relativistic limit for positive energy solutions. They are switched for negative energy solutions which represent (the absence of) the anti-particle. It is also useful to consider the ultra-relativistic limit $E \gg m$ (or $m \to 0$) in order to discuss high-energy processes and massless neutrinos.

We start with the Dirac equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi = [c\vec{\alpha}\cdot\vec{p} + mc^2\beta]\psi. \tag{1}$$

We solved it using the two-component eigenvectors of $\vec{\sigma}\cdot\vec{p}$

$$\vec{\sigma}\cdot\vec{p}\chi_+(\vec{p}) = \vec{\sigma}\cdot\vec{p}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = +|\vec{p}|\chi_+(\vec{p}), \tag{2}$$

$$\vec{\sigma}\cdot\vec{p}\chi_-(\vec{p}) = \vec{\sigma}\cdot\vec{p}\begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} = -|\vec{p}|\chi_-(\vec{p}). \tag{3}$$

The positive energy solutions are $\psi(\vec{x},t) = u_\pm(p)e^{-ip_\mu x^\mu/\hbar}$,

$$u_+(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\chi_+(\vec{p}) \\ \sqrt{\frac{E-mc^2}{2mc^2}}\chi_+(\vec{p}) \end{pmatrix}, \qquad u_-(p) = \begin{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}}\chi_-(\vec{p}) \\ -\sqrt{\frac{E-mc^2}{2mc^2}}\chi_-(\vec{p}) \end{pmatrix}, \tag{4}$$

while the negative energy solutions are $\psi(\vec{x},t) = v_\pm(p)e^{+ip_\mu x^\mu/\hbar}$,

$$v_+(p) = \begin{pmatrix} \sqrt{\frac{E-mc^2}{2mc^2}}\chi_+(\vec{p}) \\ \sqrt{\frac{E+mc^2}{2mc^2}}\chi_+(\vec{p}) \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} -\sqrt{\frac{E-mc^2}{2mc^2}}\chi_-(\vec{p}) \\ \sqrt{\frac{E+mc^2}{2mc^2}}\chi_-(\vec{p}) \end{pmatrix}. \tag{5}$$

It is easy to check that they are helicity eigenstates. We have seen before that the total angular momentum is given by $\vec{J} = \vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}$, which is conserved, while the orbital and spin angular momenta are individually not conserved. In order to seperate the spin degree of freedom, it is useful to project it along the direction of the momentum,

$$\vec{p} \cdot \vec{J} = \vec{p} \cdot \frac{\hbar}{2} \vec{\Sigma}. \tag{6}$$

The orbital angular momentum drops out and we single out the spin. The helicity is defined by

$$h = \frac{\vec{p} \cdot \vec{J}}{|\vec{p}|} = \frac{\hbar}{2} \frac{\vec{p} \cdot \vec{\Sigma}}{|\vec{p}|}. \tag{7}$$

The positive helicity $+\hbar/2$ is called "right-handed," while the negative helicity $-\hbar/2$ "left-handed." It is important to note that the helicity is frame-dependent if the particle has a (not matter how small) a finite mass. If the mass is finite, in principle you can go faster than it an look back. The momentum appears to go the opposite direction in your rest frame, while the spin remains the same. Therefore, you would observe the opposite helicity. On the other hand, if the particle is massless, you can never pass it, and hence the helicity is frame-independent.

Now we take the ultra-relativistic limit $E \gg m$. Up to the normalization factor of $\sqrt{E/mc^2}$, we find

$$u_+(p) \propto \begin{pmatrix} \chi_+(\vec{p}) \\ \chi_+(\vec{p}) \end{pmatrix}, \qquad u_-(p) \propto \begin{pmatrix} \chi_-(\vec{p}) \\ -\chi_-(\vec{p}) \end{pmatrix}, \tag{8}$$

$$v_+(p) \propto \begin{pmatrix} \chi_+(\vec{p}) \\ \chi_+(\vec{p}) \end{pmatrix}, \qquad v_-(p) \propto \begin{pmatrix} -\chi_-(\vec{p}) \\ \chi_-(\vec{p}) \end{pmatrix}. \tag{9}$$

They have simplified dramatically. In particular, by introducing a matrix

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{10}$$

we find they are all eigenstates of this matrix,

$$\gamma_5 u_\pm(p) = \pm u_\pm(p), \qquad \gamma_5 v_\pm(p) = \pm v_\pm(p). \tag{11}$$

The eigenvalue of γ_5 is called "chirality," originating from a Greek word that means "hand." The name suggests that it is closely related to the handedness, namely the helicity of the particle. Indeed, what we see here is that, in the limit of $E \gg m$ or massless limit, the chirality and the helicity are in one-to-one correspondence.

The chirality is a good quantum number if the particle is massless. This is because γ_5 commutes with $\vec{\alpha}$, while anti-commutes with β. Therefore,

$$i\hbar\frac{d}{dt}\gamma_5 = [\gamma_5, H] = 2mc^2\gamma_5\beta, \tag{12}$$

and hence the chirality is conserved only for a massless fermion. On the other hand, you can show that γ_5 is Lorentz-invariant. This point is consistent with the previous discussion that the helicity is frame-independent only for a massless particle. If the particle is massless, the helicity is in one-to-one correspondence with the chirality, which is Lorentz-invariant.

2 Complete Set of Dirac Matrices

It is useful to make the Dirac equation manifestly Lorentz-covariant. One good way to start is to rewrite it in the following manner

$$\left(i\hbar\frac{\partial}{c\partial t} + i\hbar\vec{\alpha}\cdot\vec{\nabla} - mc\beta\right)\psi = 0. \tag{13}$$

Because the mass is a Lorentz scalar, it is natural to multiply the equation by β to make it purely a constant (recall $\beta^2 = I$),

$$\left(i\hbar\beta\frac{\partial}{c\partial t} + i\hbar\beta\vec{\alpha}\cdot\vec{\nabla} - mc\right)\psi = 0. \tag{14}$$

This form suggests that $\gamma^\mu = (\beta, \beta\vec{\alpha})$ transforms as a Lorentz vector. Then the equation has the manifestly Lorentz-covariant form

$$(i\gamma^\mu\partial_\mu - mc)\,\psi = 0. \tag{15}$$

The explicit forms for gamma-matrices are easily obtained from β and $\vec{\alpha}$,

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{16}$$

Note that the gamma matrices all anti-commute,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{17}$$

Moreover, γ_5 anti-commutes with all other gamma matrices,

$$\{\gamma_5, \gamma^\mu\} = 0. \tag{18}$$

Because these are four-by-four matrices, it is a useful question to ask (and we will use it in the theory of weak interactions) what the complete set of Dirac matrices is. Obviously, there are sixteen linearly independent matrices. They are grouped together according to their Lorentz transformation properties as

$$1 \qquad \text{Scalar} \tag{19}$$

$$\gamma^\mu \qquad \text{Vector} \tag{20}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad \text{Tensor} \tag{21}$$

$$\gamma^\mu\gamma_5 \qquad \text{Axial-Vector} \tag{22}$$

$$\gamma_5 \qquad \text{Pseudo-Scalar.} \tag{23}$$

Note that the tensor has only six (${}_4C_2 = 6$) matrices becuase of the anti-symmetry of two indices. The total number is $1 + 4 + 6 + 4 + 1 = 16$ as desired. The axial-vector differs from the vector in terms of the parity, and so does pseudo-scalar from scalar. We will discuss parity in the next section.

3 Parity

How do we define parity on the Dirac wavefunction? The key is to make sure that the Dirac equation remains the same before and after the parity. We also include the electromagnetic potential to help us interpret the physical meaning of it.

The Dirac equation in its covariant form in the presence of the electromagnetic potential is

$$\left(i\gamma^\mu(\partial_\mu + i\frac{e}{c}A_\mu(\vec{x}, t)) - mc\right)\psi(\vec{x}, t) = 0. \tag{24}$$

What we would like to achive is to flip the space $\vec{x} \to -\vec{x}$. Because the equation is supposed to hold at any position in space, we can simply subsitute $-\vec{x}$ into $\vec{x}$, and we find

$$\left(i\gamma^0(\partial_0 + i\frac{e}{c}A_0(-\vec{x}, t)) + i\gamma^i(-\partial_i + i\frac{e}{c}A_i(-\vec{x}, t)) - mc\right)\psi(-\vec{x}, t) = 0. \tag{25}$$

Here I used the fact that the derivative also changes the sign under this substitution. The question is to bring this back to the form as close as possible to the original equation. Thanks to the anti-commutation property of gamma matrices, it is easy to see that $\gamma^0\gamma^i\gamma^0 = -\gamma^i$. Therefore,

$$\left(i\gamma^0(\partial_0 + i\frac{e}{c}A_0(-\vec{x},t)) + i\gamma^0\gamma^i\gamma^0(\partial_i - i\frac{e}{c}A_i(-\vec{x},t)) - mc\right)\psi(-\vec{x},t) = 0. \tag{26}$$

By multiplying the equation by γ^0 from the left,

$$\left(i\gamma^0(\partial_0 + i\frac{e}{c}A_0(-\vec{x},t)) + i\gamma^i(\partial_i - i\frac{e}{c}A_i(-\vec{x},t)) - mc\right)\gamma^0\psi(-\vec{x},t) = 0. \tag{27}$$

Therefore, $\gamma^0\psi(-\vec{x},t)$ almost satisfies the same equation as before, except the sign of the vector potential. Indeed, we have forgotten to change the sign of the vector potential! Under parity, the electric field should change its sign, while $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$. Therefore the vector potential should change the sign under the parity. It is also consistent with the axial-vector nature of the magnetic field, $\vec{B} = \vec{\nabla} \times \vec{A}$ which changes the sign twice. Now that we remember this, the correct parity-transformed Dirac equation is

$$\left(i\gamma^0(\partial_0 + i\frac{e}{c}A_0(-\vec{x},t)) + i\gamma^i(\partial_i + i\frac{e}{c}A_i(-\vec{x},t)) - mc\right)\gamma^0\psi(-\vec{x},t) = 0, \tag{28}$$

which is of exactly the same form as the original equation. In summary, the parity transformation is given by

$$\psi(\vec{x},t) \rightarrow \gamma^0\psi(-\vec{x},t), \tag{29}$$
$$A_0(\vec{x},t) \rightarrow A_0(-\vec{x},t), \tag{30}$$
$$A_i(\vec{x},t) \rightarrow -A_i(-\vec{x},t). \tag{31}$$

The immediate consequence of this is the parity eigenvalues of the particle and anti-particle at rest. For the particle at rest $\vec{p} = 0$, $u_\pm(p)$ are non-vanishing only in the first two components, while for the anti-particle at rest we use the negative-energy solutions $v_\pm(p)$ which are non-vanishing only in the lower two components. Because $\gamma^0 = \mathrm{diag}(1,1,-1,-1)$, it follows that the particle at rest has parity eigenvalue $+1$, while the anti-particle at rest has parity eigenvalue -1. Of course, the distinction between particle and anti-particle is a convention. If you change your convention that you call positrons as particles and electrons anti-particles, even though it is awfully inconvenient (we live in the world of anti-matter in that case!), the positron has the even parity while the electron odd. It may sound troublesome that the parity eigenvalue is convention dependent, but it turns out the physical quantities won't.

To do the reality check of the parity eigenvalue we just found, let us consider the charged pion $\pi^+ = u\bar{d}$. There is no orbital angular momentum $L = 0$, and the spins are anti-aligned $S = 0$ to form spin-zero meson. Under parity, u, being a particle, returns a positive sign, while $\bar{d}$, being an anti-particle, returns a negative sign. Therefore the pion has the odd parity 0^-, as we talked about already several times. Note that even if you switch to the other convention, you still get the overall minus sign. The spin parallel counter part $\rho^+ = u\bar{d}$ also has the same odd parity: 1^-.

In general, if the meson has the orbital angular momentum L between the quark and the anti-quark, it gives the parity $(-1)^L$ from the spherical harmonics. In addition, either quark or anti-quark, depending on your convention, gives the minus sign, and hence the parity eigenvalue of the meson is $(-1)^{L+1}$. Using the parity and the spin of the meson, you can figure out the orbital angular momentum. For example, when $L = 1$, $S = 0$, the spin must be $J = 1$, while when $S = 1$ for the same L, the spin can be $J = 0, 1, 2$ (addition of the angular momentum 1 and 1). All of them must share the same positive parity. Indeed, the $I = 1$ mesons $b_1(1235)$ are the $L = 1$, $S = 0$ case, and $a_0(1450)$, $a_1(1260)$, and $a_2(1320)$ are the $L = 1$, $S = 1$ cases. They are all parity even. Look at `http://pdg.lbl.gov/2002/quarkmodrpp.pdf` for a table of mesons, their quantum numbers, and their spectroscopic classifications.

It is also important to note that γ_5 is parity-odd, and hence pseudo-scalar. It follows simply because γ_5 anti-commutes with γ^0.

4 Charge Conjugation

Another important symmetry is the interchange of particles and anti-particles. The world of anti-matter would look just the same as ours, made up of anti-nucleons and positrons forming anti-atoms. Indeed, Athena experiment at CERN (`http://athena.web.cern.ch/athena`) has produced many anti-hydrogen atoms. The overall switch of particles and anti-particles is called "charged conjugation" C, because that would flip the charges such as electric charge, strangeness, baryon number and so on.

What we want to do then is to find a way to change the Dirac equation such that the positive- and negative-energy solutions are interchanged while the form of the equation remains the same. The good guess is that it involves a complex conjugation so that $e^{-iEt/\hbar}$ becomes $e^{+iEt/\hbar}$ and vice versa. Given this guess, let us take the complex conjugate of the whole equation,

$$\left(-i\gamma^{\mu *}(\partial_\mu - i\frac{e}{c}A_\mu(\vec{x},t)) - mc\right)\psi^*(\vec{x},t) = 0. \tag{32}$$

The obvious problem is that $-i\gamma^{\mu *}$. We have to bring them back to $i\gamma^\mu$.

Recall that $\gamma^{0,1,3}$ are all real, while γ^2 is imaginary. The only one that changes its sign under the complex conjugation is therefore γ^2. Using the anti-commutation of gamma matrices, we can write the general equation

$$(i\gamma^2)\gamma^{\mu *}(i\gamma^2) = -\gamma^\mu. \tag{33}$$

Note that $i\gamma^2$ is hermitean and unitarity. By multiplying the equation by $(i\gamma^2)$ from the left, we find

$$\left(i\gamma^{\mu}(\partial_\mu - i\frac{e}{c}A_\mu(\vec{x},t)) - mc\right)i\gamma^2\psi^*(\vec{x},t) = 0. \tag{34}$$

Therefore the equation is the same as before, except that the sign of the electromagnetic potential is the opposite. Well, under the charge conjugation, we should have also flipped the sign of the electromagnetic potential! Doing so, the Dirac equation maintains exactly the same form as before. In other words, the electromagnetic potential is odd under the charged conjugation, and the photon has the odd eigenvalue under C. You can also check that the positive- and negative-energy solutions are interchanged by $\psi(\vec{x},t) \to i\gamma^2\psi^*(\vec{x},t)$.

It is interesting to apply the charge conjugation to mesons. Some particles are eigenstates of the charge conjugation, namely that the particle and its anti-particle are the same. The photon was indeed a good example, $C|\gamma\rangle = -|\gamma\rangle$. In order for this to be possible the particle must be neutral under all charges. Another such example is π^0. Its wave function consistent with $I=1$ and $S=0$ is

$$|\pi^0\rangle = |\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\rangle = \frac{1}{2}|u^\uparrow\bar{u}^\downarrow - u^\downarrow\bar{u}^\uparrow - d^\uparrow\bar{d}^\downarrow + d^\downarrow\bar{d}^\uparrow\rangle. \tag{35}$$

Under the charge conjugation, it becomes

$$C|\pi^0\rangle = \frac{1}{2}|\bar{u}^\uparrow u^\downarrow - \bar{u}^\downarrow u^\uparrow - \bar{d}^\uparrow d^\downarrow + \bar{d}^\downarrow d^\uparrow\rangle. \tag{36}$$

We want to go back to the original order of quark and anti-quark, but being fermions, their interchange produces a minus sign.

$$C|\pi^0\rangle = \frac{1}{2}| - u^\downarrow\bar{u}^\uparrow + u^\uparrow\bar{u}^\downarrow + d^\downarrow\bar{d}^\uparrow - d^\uparrow\bar{d}^\downarrow\rangle. \tag{37}$$

This is the same as the original wave function, and hence $C|\pi^0\rangle = +|\pi^0\rangle$. Note that the minus sign from the Fermi statistics was compensated by the anti-symmetry of the spin wave function. In other words, the overall sign was determined by $(-1)^S$. If we apply the same method to $|\rho^0\rangle = |\frac{1}{\sqrt{2}}(u^\uparrow\bar{u}^\uparrow - d^\uparrow\bar{d}^\uparrow)\rangle$, we find

$$C|\rho^0\rangle = C|\frac{1}{\sqrt{2}}(u^\uparrow\bar{u}^\uparrow - d^\uparrow\bar{d}^\uparrow)\rangle = |\frac{1}{\sqrt{2}}(\bar{u}^\uparrow u^\uparrow - \bar{d}^\uparrow d^\uparrow)\rangle = |-\frac{1}{\sqrt{2}}(u^\uparrow\bar{u}^\uparrow - d^\uparrow\bar{d}^\uparrow)\rangle = -|\rho^0\rangle. \tag{38}$$

When applied to mesons with non-zero orbital angular momentum, there is an additional sign of $(-1)^L$, and hence the overall charged conjugation is given by $(-1)^{L+S}$. The table of mesons indeed shows the different charged conjugation quantum numbers between $S=0$ and $S=1$ states of the same flavor compositions and the same orbital angular momenta.

The reality check here is that $\pi^0 \to \gamma\gamma$ decay is allowed as observed. The initial state is even under C, while the final state has two C-odd photons, and hence again even under C. On the other hand, $\rho^0 \to \gamma\gamma$, even though it preserves all other conserved quantities, is forbidden because of the charge conjugation invariance. It can decay into three photons (extremely rare) but not into two.

The same analysis applies to the positronium, the bound state of e^+ and e^- by a simple Coulomb attraction. After separating the center-of-mass motion, the system is the same as the hydrogen atom of reduced mass $m_e/2$. The ortho and para positronia correspond to $S=0$ and $S=1$ combinations, which are split only by the hyperfine interaction. But they decay in completely different ways. $S=0$ state can decay into two photons quickly. However $S=1$ state must decay into three photons, and the decay rate is suppressed by three powers of the fine-structure constant. This difference can be exploited in the condensed matter experiments using the positron annihilation. Using the emission of photons, one can study both the spatial and spin distribution of electrons in a given material.