

# 129A Lecture Notes

## Weak Interactions II

### 1 Glashow–Weinberg–Salam Theory

We have seen that the weak interaction happens “pretty much” on the isospin lowering or raising operator, except for small admixture of the strange quark. On the side of leptons, they are strictly between  $e$  and  $\nu_e$ , or  $\mu$  and  $\nu_\mu$ . Moreover, the strength of the weak interaction is universal, *i.e.*, it is described by the same Fermi constant for any processes. This point strongly suggests that the weak interaction is caused by the similar mechanism as the electromagnetism or quark-gluon theory of the strong interaction, where the strength of the force is determined by the overall coupling constant and the charges (or matrices) of given particle type. Based on this motivation, Sheldon Glashow in 1962, and later Steven Weinberg and Abdus Salam, tried to formulate the theory of the weak interaction in terms of the same type of theory, namely gauge theory.

Recall the case of the gluon. It is based on three colors, and you demand that you can freely rotate among three colors. This defines three-by-three unitarity rotations, given by the group  $SU(3)$  (after removing the overall phase part). There are eight generators for this group, given by Gell-Mann’s lambda matrices (with a factor of  $1/2$ ). There are eight gluons, each of which couples to each generator. When the generator acts on a quark, it may change its color. In the same way, we look at the “weak isospin”  $SU(2)$ . We distinguish it from the “strong isospin” which is between up and down with no admixture of strange, and has nothing to do with leptons. We put together doublets

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}. \quad (1)$$

Under the two-by-two unitarity rotation without the overall phase, the  $SU(2)$  group, there are three generators given by the three Pauli matrices  $\tau_1, \tau_2, \tau_3$  (with a factor of  $1/2$ ). Accordingly, we postulate three  $W$ -bosons,  $W_1, W_2, W_3$ . Obviously  $W$  stands for “weak.” When they interact with doublets,

they produce a set of amplitudes given by

$$(\bar{\nu}_\mu, \bar{\mu})(g\vec{W} \cdot \frac{\vec{\tau}}{2}) \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} = (\bar{\nu}_\mu, \bar{\mu}) \frac{g}{2} \begin{pmatrix} W_3 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}. \quad (2)$$

We introduce the notation  $W^+ = (W_1 - iW_2)/\sqrt{2}$ ,  $W^- = (W_1 + iW_2)/\sqrt{2}$ , so that it becomes

$$(\bar{\nu}_\mu, \bar{\mu}) \frac{g}{2} \begin{pmatrix} W_3 & \sqrt{2}W^+ \\ \sqrt{2}W^- & -W_3 \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}. \quad (3)$$

The same set is there also for other doublets. Therefore, there are vertices such as  $\mu^- \rightarrow \nu_\mu W^-$ ,  $\nu_e \rightarrow e^- W^+$ , etc. All these vertices come with the universal strength  $g$ , analogous to  $e$  in the case of electromagnetism.

This is very nice. The muon decay, for example, happens because we use the vertex  $\mu^- \rightarrow \nu_\mu W^-$ , where  $W^-$  is virtual. This virtual  $W^-$  has to materialize quickly, such as  $W^- \rightarrow e^- \bar{\nu}_e$ . If you work out numbers, the Fermi constant is then given by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}. \quad (4)$$

You get two factors of  $g$  because you use two vertices, and the amplitude is suppressed by the heaviness of the  $W$ -boson. The same can be said about the neutron beta decay, where one of the down-quarks in the neutron emits  $d \rightarrow u W^-$ , followed quickly by  $W^- \rightarrow e^- \bar{\nu}_e$ . Because all these are based on the same interaction, no wonder why they come out universal (modulo the mixing of the strange quark). Hurray! We got the theory of the weak force!

But it is a little bit too early to celebrate a victory. There are several tricky points which we haven't seen in other forces we have to deal with.

One is that the weak force is of  $V - A$ , and hence only the chirality left  $\gamma_5 = -1$  component couples to it. The way to achieve it is to assume that the right-handed chirality particles are singlets under the weak isospin, so that  $W$ -bosons don't "see" them, in the same way that the photons don't see neutrinos because of the lack of electric charge.  $W$ -bosons couple to "doubletiness," and they do not interact with singlets. Therefore, we have to accept that the particles are organized as

$$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}, u_R, d_R, \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, e_R, \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix}, \mu_R. \quad (5)$$

I didn't put the subscript  $L$  on neutrinos because they are all left-handed. It looks very strange, but this is the only way we can explain the  $V - A$  nature of the weak interaction. In other words, we treat left-handed and right-handed particles as *different particles*. Of course this raises the concern how come they can be massive, because the chirality is not a good quantum number for massive fermions as we discussed before. We will come back to the issue of the mass later, and let us forge ahead for the moment.

The next tricky point is that we have to somehow get photon out of this. The electric charge does not commute with the isospin raising or lowering operators, so that it must be a part of the group we introduce to explain the forces. The natural candidate here is of course  $W_3$ . However, we can see immediately that this won't work.  $W_3$  couples to neutrinos! We do not want photons to interact with neutrinos because they are electrically neutral. We are still missing something.

What it means is that the  $SU(2)$  is not enough to explain the weak interaction and the electromagnetism at the same time. We need something more. And we would like to do so without any more proliferation of particles.

The only way to get out of this dilemma is to introduce another charge, called "weak hypercharge" (this is another bad example of reusing the same name to mean something completely different). We also introduce a new spin one hypercharge boson  $B$ , similarly as the photon (that is why it is  $B$ , next to  $A$  for the vector potential). To get started we assign the charge  $-1/2$  to the lepton doublets. We could have chosen a different charge, but this turns out to be convenient. Then the set of amplitudes we deal with includes the  $B$ -boson,

$$(\bar{\nu}_\mu, \bar{\mu})(g\vec{W}\cdot\frac{\vec{\tau}}{2} + g'B\frac{-1}{2}) \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} = (\bar{\nu}_\mu, \bar{\mu})\frac{1}{2} \begin{pmatrix} gW_3 - g'B & g\sqrt{2}W^- \\ g\sqrt{2}W^+ & -gW_3 - g'B \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}. \quad (6)$$

We have two neutral spin one bosons,  $B$  and  $W_3$ . No matter what we do, we end up with two neutral spin one bosons, and one of them must be our good-old photon. Then the combination that couples to the neutrino had better be the other one. In other words, there must be a new spin one boson, called  $Z$ ,<sup>1</sup> given by

$$\sqrt{g^2 + g'^2}Z = gW_3 - g'B. \quad (7)$$

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<sup>1</sup>This is another arrogant name next to  $\Omega^-$ . This is the "last" boson. Later on, when people started to discuss grand unified theories, we had to call new bosons  $X$  and  $Y$ .

The prefactor is the normalization factor. The photon must be the orthogonal combination,

$$\sqrt{g^2 + g'^2}A = g'W_3 + gB. \quad (8)$$

Instead of always referring to the coupling constants  $g$  and  $g'$ , it is convenient to introduce the “weak mixing angle,”<sup>2</sup>

$$\theta_W = \sin^{-1} \frac{g'}{\sqrt{g^2 + g'^2}}. \quad (9)$$

This allows us to rewrite the  $Z$  and photon as simple as

$$Z = W_3 \cos \theta_W - B \sin \theta_W, \quad (10)$$

$$A = W_3 \sin \theta_W + B \cos \theta_W. \quad (11)$$

Not only that we mix quarks, we also have to mix gauge bosons.

Once we have done this, the interaction of the photon and the  $Z$ -boson to the muon (or electron) is fixed uniquely,

$$\begin{aligned} \frac{1}{2}(-gW_3 - g'B) &= \frac{1}{2}(-g(Z \cos \theta_W + A \sin \theta_W) - g'(-Z \sin \theta_W + A \cos \theta_W)) \\ &= \frac{1}{2}(-g \cos \theta_W + g' \sin \theta_W)Z - \frac{1}{2}(g \sin \theta_W + g' \cos \theta_W)A. \end{aligned} \quad (12)$$

The coupling to the photon must be  $e = -|e|$ ,

$$|e| = \frac{1}{2}(g \sin \theta_W + g' \cos \theta_W) = \frac{gg'}{\sqrt{g^2 + g'^2}}. \quad (13)$$

In other words, one combination of two new coupling constants we introduced is already measured, and the unknown parameter is just  $\theta_W$ ,

$$g = \frac{|e|}{\sin \theta_W}, \quad g' = \frac{|e|}{\cos \theta_W}. \quad (14)$$

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<sup>2</sup>I grew up with people who called it “Weinberg angle.” But Weinberg’s paper actually does not introduce this angle, but rather uses  $g$  and  $g'$  throughout. It was Glashow who introduced this angle earlier. But  $\theta_W$  is a wide-spread notation and we can’t call it Glashow angle anymore. Recently the name “weak mixing angle” is used more widely that solves this problem.

Then the coupling of the left-handed muon (or electron) to the  $Z$ -boson is also determined,

$$\frac{1}{2}(-g \cos \theta_W + g' \sin \theta_W) = \frac{|e|}{\sin \theta_W \cos \theta_W} \left( -\frac{1}{2} + \sin^2 \theta_W \right). \quad (15)$$

In general, any particle would couple to the combination  $gI_3W_3 + g'YB$ . Here,  $I_3$  is the third component of the isospin  $\pm\frac{1}{2}$  for doublets and 0 for singlets.  $Y$  is the “weak hypercharge,” which we would like to determine now. By rewriting this combination using the  $Z$ -boson and the photon, we find

$$\begin{aligned} & gI_3(Z \cos \theta_W + A \sin \theta_W) + g'Y(-Z \sin \theta_W + A \cos \theta_W) \\ &= \frac{|e|}{\sin \theta_W \cos \theta_W} (I_3 \cos^2 \theta_W - g'Y \sin^2 \theta_W)Z + |e|(I_3 + Y)A. \end{aligned} \quad (16)$$

The electric charge of the particle is then given by  $Q = I_3 + Y$ . This helps us to determine what we have to take for hypercharges for individual particles. This is the “weak” analogue of Gell-Mann–Nishijima relation. The only possible hypercharge assignment is

$$\left( \begin{array}{c} u_L \\ d'_L \end{array} \right)^{+1/6}, u_R^{+2/3}, d_R^{-1/3}, \left( \begin{array}{c} \nu_e \\ e_L \end{array} \right)^{-1/2}, e_R^{-1}, \left( \begin{array}{c} \nu_\mu \\ \mu_L \end{array} \right)^{-1/2}, \mu_R^{-1}. \quad (17)$$

The hypercharges are shown as superscripts.

Because we have two types of gauge bosons, one coupled to Pauli matrices ( $SU(2)$ ) and the other to numbers (one-by-one hermitian matrices are generators, and hence their exponentials are one-by-one-unitarity:  $U(1)$ ), this theory is called  $SU(2) \times U(1)$ . The hypercharge assignments look very bizarre, but apart from that, everything works fine. We get the  $V - A$  interaction, we understand the universality of the weak interaction, and we correctly reproduced the photon. On the other hand, we now predict a new force, mediated by the  $Z$ -boson, called “neutral-current weak interaction.” We do not know the new parameter, the weak mixing angle yet, and we do not know the masses of the  $W$ - and  $Z$ -boson yet. But the combination  $G_F/\sqrt{2} = g^2/8m_W^2 = e^2/8m_W^2 \sin^2 \theta_W$  is already fixed. Once we know the weak mixing angle, we will know the mass of the  $W$ -boson.

## 2 Neutral-Current Weak Interaction

The coupling of the  $Z$ -boson is determined completely by the weak isospin and weak hypercharge assignments discussed in the previous section. Let me emphasize again that there was no freedom at all in these charge assignments, no matter how bizarre the outcome may look. Just knowing that there is  $V - A$  charged current weak interaction and we know the electric charges, they came out uniquely. Then the neutral-current weak interaction is also determined uniquely.

We have seen that the coupling of the  $Z$ -boson is given by

$$\frac{|e|}{\sin \theta_W \cos \theta_W} (I_3 \cos^2 \theta_W - g' Y \sin^2 \theta_W) Z. \quad (18)$$

It is more customary to rewrite it a little bit further using  $Q = I_3 + Y$  and  $g_Z = |e|/\sin \theta_W \cos \theta_W$ ,

$$g_Z (I_3 - Q \sin^2 \theta_W). \quad (19)$$

$I_3 = \pm \frac{1}{2}$  for left-handed particles, and  $I_3 = 0$  for right-handed particles. One important point is that even right-handed particles do participate in the neutral-current weak interaction, while they don't in the charged-current weak interaction. We will see that  $\sin^2 \theta_W \simeq 0.23$ .

The neutral-current weak interaction was found in neutrino experiments. Gargamelle collaboration build a big bubble chamber filled with liquid hydrogen. The advantage of this was that the detector (bubble chamber) was also a target (hydrogen) at the same time, and it allowed a large volume. Any neutrino experiment needs this type of combination (detector and target at the same time with a large volume) because of the small probabilities for neutrinos to interact. They specifically looked for the reaction  $\bar{\nu}_\mu e^- \rightarrow \bar{\nu}_\mu e^-$ . The point is that the charged-current weak interaction is impossible for this initial state. If  $\bar{\nu}_\mu$  emits a virtual  $W^-$  and becomes  $\mu^+$ , the virtual  $W^-$  must be quickly absorbed by the electron, but of course it can't be with two units of charges. The only way for the electron to be knocked out from the atomic orbit by  $\bar{\nu}_\mu$  would be through a new type of interaction. See Cahn–Goldhaber, Chapter 12 for the story of the discovery in 1973.

Many other neutrino scattering experiments were done to study the neutral-current weak interactions. Most of them used the scattering of neutrinos off protons, but such experiments suffer from the lack of detailed knowledge in the structure of the proton and led to confusing results. At some point, there

was a serious discrepancy between the prediction of  $SU(2) \times U(1)$  theory and the data, called “high- $y$  anomaly.” But this problem went away by about 1978. One of the decisive experiments was done at SLAC, the scattering of polarized electrons of deuterium target. You look for the difference in the rate of scattering rate between two polarizations. The scattering is mostly due to the exchange of a virtual photon, which is common to both polarizations, but a small effect of the virtual  $Z$ -boson exchange interferes with the photon-exchange amplitude. The coupling of the left-handed electron is proportional to  $g_Z(I_3 - Q \sin^2 \theta_W \sim -0.27)$ , while that of the right-handed coupling to  $g_Z(-Q \sin^2 \theta_W \sim +0.23)$ . Therefore, the relative sign between two amplitudes is the opposite. The difference in the cross section between two polarizations of the electron is proportional to the difference between the left- and right-handed coupling  $g_Z I_3 = e/2 \sin \theta_W \cos \theta_W$ . Many measurements of  $\sin^2 \theta_W$  converged, leading to Nobel prize to Glashow, Weinberg and Salam in 1979.

### 3 Charm

One problem Glashow struggled with was the question of so-called flavor-changing neutral current. We have assigned the up quark and a linear combination  $d'_L \equiv d_L \cos \theta_C + s_L \sin \theta_C$  as a doublet. According to the discussion in the previous section, this combination would have the neutral-current weak interaction,

$$\bar{d}'_L \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) d'_L Z \quad (20)$$

The problem is this. If you write this out in terms of the mass eigenstates  $d_L$  and  $s_L$ , you find a new interaction that changes the flavor,

$$\bar{d}_L \cos \theta_C \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) s_L \sin \theta_C Z \quad (21)$$

and its complex conjugate. With this interaction, the neutral kaon  $K^0(d\bar{s})$  can transform to  $\bar{K}^0(s\bar{d})$  at a far too large rate. One of the diagrams is that the  $d$  emits a virtual  $Z$  and become  $s$ , and the  $Z$  is absorbed by the  $\bar{s}$  which becomes  $\bar{d}$ . Because the  $Z$  boson is heavy (91 GeV), this force is very short-ranged, and happens only when  $d$  and  $\bar{s}$  basically come to the same point within the size of the kaon. The size of the wave function at the same point can be measured in the process  $K^+ \rightarrow \mu^+ \nu_\mu$ , and using the isospin

symmetry, it determines the wave function for  $K^0$ . It is characterized by a “kaon decay constant”  $f_K \sim 160$  MeV. (See “Pseudoscalar-Meson Decay Constants” [http://pdg.lbl.gov/2002/decaycons\\_s808.pdf](http://pdg.lbl.gov/2002/decaycons_s808.pdf) from Particle Data Group for more details.) Then this process would induce

$$\Delta m_K^2 \sim \frac{g_Z^2}{m_Z^2} \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right)^2 \sin^2 \theta_C \cos^2 \theta_C f_K^2 m_K^2. \quad (22)$$

This is many orders of magnitudes larger than the observed mass splitting and is completely unacceptable. Because this is a neutral-current weak interaction that changes flavor, it is called FCNC (flavor-changing neutral current) process.

Glashow, Iliopoulos, and Maiani pointed out that this problem of flavor-changing coupling of the  $Z$ -boson can be avoided if there exists the charm quark in 1970. The point is that the charm quark belongs to a doublet together with  $s' = -d \sin \theta_C + s \cos \theta_C$ , the orthogonal combination from  $d'$ . Then the  $Z$ -coupling to  $s'$  also gives a flavor-changing neutral current

$$\bar{d}_L (-\sin \theta_C) \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) s_L \cos \theta_C Z. \quad (23)$$

The important point is that this amplitude precisely cancels the contribution from  $d'$  because of the orthogonality condition. The existence of the charm quark was speculated before them, based on the correspondence between the lepton (two generations had been known since Neddermeyer–Anderson) and the quarks (only up, down, and strange had been known until  $J/\psi$ ). However, they were the first to point out a *necessity* of the charm quark.

It turns out that the absence of the dangerous  $Z$  coupling is not enough to avoid a too-large  $K^0$ - $\bar{K}^0$  mixing. There is a second-order weak interaction called “the box diagram,” where the down and anti-strange quarks in the initial  $K^0$  exchange a virtual  $W$  and become up- or charm-quarks, which again exchange a virtual  $W$  to become  $\bar{K}^0$ . If the up and charm quarks have the same mass, four diagrams with  $u\bar{u}$ ,  $u\bar{c}$ ,  $c\bar{u}$ , and  $c\bar{c}$  cancel exactly again because of the orthogonality condition. However, the charm quark had not been observed and it has be heavier than the up quark if it exists. Once the masses are different, the four diagrams no longer exactly cancel. Therefore, the four diagrams gives a contribution of approximately  $\frac{1}{16\pi^2} G_F^2 m_c^2 \cos^2 \theta_C \sin^2 \theta_C f_K^2 m_K^2$  (the prefactor is a typical factor for diagrams with one loop).

It was Mary K Gaillard in our department and Ben Lee who realized that one can set an upper limit on the charm quark mass from this argument. They said  $m_c \sim 1.5$  GeV would give the correct size of  $K^0-\bar{K}^0$  in 1974, which was dramatically confirmed later (November) that year.

## 4 $\tau$ -lepton

Soon after the November revolution, SLAC SPEAR was inching up its energy looking for more resonances and  $D$ -mesons. Martin Perl on SLAC-LBL collaboration noticed that there are events with one electron and one muon with large missing energy/momentum. They were called “anomalous  $e \mu$  events,” which was published in 1975. The interpretation was that this was another charged lepton, called  $\tau$  for “tertiary” (the third). This claim had not been believed for a few years. I’ve heard that it was too much for people to accept that there is another new elementary particle right after the charm quark was discovered. But it was eventually established, and Perl was awarded Nobel prize for the discovery of the first third-generation particle before Lederman’s Upsilon (bottom anti-bottom bound state) in 1977 and the top quark in 1995.

How do we understand the anomalous  $e \mu$  events? The property of the  $\tau$  lepton is exactly the same as the electron and the muon. Therefore it can decay via the charged-current weak interaction by the exchange of a virtual  $W$ . It is easy to see that both  $\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e$  and  $\nu_\tau \mu^- \bar{\nu}_\mu$  are possible.

The  $\tau$  can also decay into hadronic final state. The  $\tau^-$  emits a virtual  $W^-$ , and  $W^-$  can then materialize into  $d\bar{u}$  (with strength  $\cos^2 \theta_C$ ),  $s\bar{u}$  (with strength  $\sin^2 \theta_C$ ), but not enough energy into charm. Produced quarks of course have to hadronize. The event Fig. 1 shows the decay  $\tau^- \rightarrow \pi^+ \pi^- \pi^- \nu_\tau$ .

Because all three generations of charged leptons share exactly the same quantum numbers, they couple to the  $W$ -boson in the same way: universality. You can see many tests of lepton universality in Anthony Pich, “Lepton Universality,” <http://arXiv.org/abs/hep-ph/9701263>.

The mass of the  $\tau$  had been measured precisely at Beijing Electron-Positron Collider (BEPC) by Beijing Spectrometer (BES) collaboration. You scan the energy region around  $2m_\tau$  and count the number of  $\tau$  events.  $\tau$  pairs cannot be produced below  $2m_\tau$ , while their production rate rises approximately as  $\beta = \sqrt{1 - m_\tau^2/E^2}$  where  $E$  is the energy of the beam. The data in Fig. 2 clearly shows such a rise in the production rate, and can be

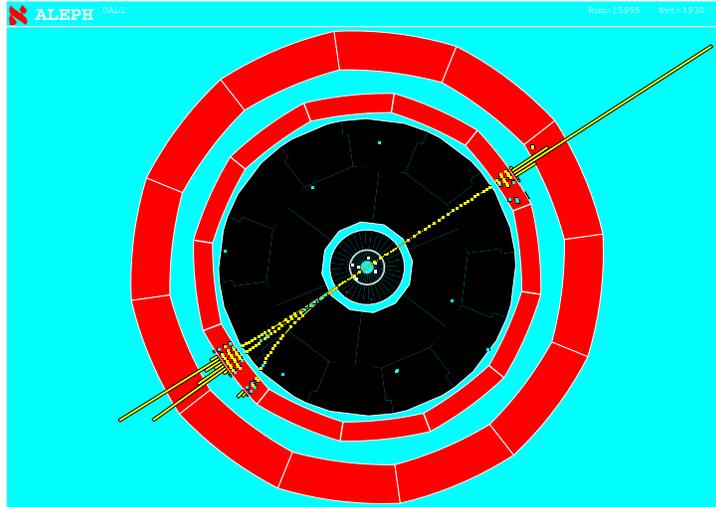


Figure 1: Pair production of tau leptons from electron-positron annihilation. One tau decays into an electron and two neutrinos, where the electrons showers in the first layer of calorimeter. The other tau decays into three pions and a neutrino where the pions are absorbed in the calorimeter.

fitted to determine  $m_\tau = 1776.96_{-0.21}^{+0.18+0.25}$  MeV.

## 5 $W$ and $Z$ Bosons

### 5.1 Discovery at $Spp\bar{p}S$

As we will see later, the  $W$  and  $Z$  boson masses are predicted in the minimal Standard Model to be  $m_W = \frac{1}{2}gv$ ,  $m_Z = \frac{1}{2}g_Zv$ , where  $v = (\sqrt{2}G_F)^{-1/2} = 250$  GeV. Once  $\sin^2\theta_W$  was measured from the neutral-current experiments, the mass of  $W$  and  $Z$  were predicted:  $m_W \sim 80$  GeV,  $m_Z \sim 90$  GeV. The masses were so much heavier than any other particles talked about before. The heaviest elementary particle seen by this point was the bottom quark, about 5 GeV. Clearly a new accelerator with an unprecedented energy was needed.

They were discovered in CERN proton anti-proton collider called  $Spp\bar{p}S$ . The main technical obstacle behind such a machine was to produce enough anti-protons, and “cool” them to small beams that can be put in accelerators. The cooling technique called stochastic cooling was developed based on the

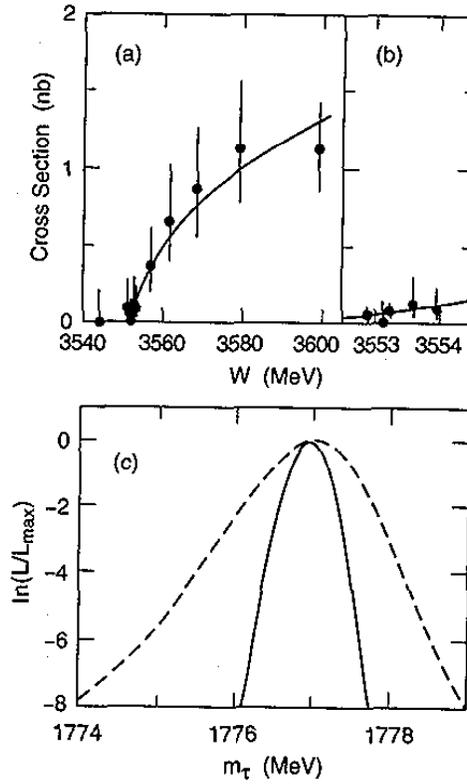


Figure 2: (a) The c.m. energy dependence of the  $\tau^+\tau^-$  cross section. (b) An expanded version of (a), in the immediate vicinity of  $\tau^+\tau^-$  threshold. (c) The solid curve shows the dependence of the logarithm of the likelihood function on  $m_\tau$ , compared to that from an older work in the dashed line. Taken from “Measurement of the mass of the  $\tau$  lepton,” BES collaboration, *Phys. Rev. D* **53**, 20–34 (1996), [http://cornell.mirror.aps.org/abstract/PRD/v53/i1/p20\\_1](http://cornell.mirror.aps.org/abstract/PRD/v53/i1/p20_1)

idea by Simon van der Meer. There was a proton synchrotron *SpS* at CERN already, and the reconfiguration of the accelerator complex to look for  $W$  and  $Z$  bosons was pushed along the vision by Carlo Rubbia.

The idea was to use up-quark in proton and anti-down-quark in anti-proton (or down and anti-up) to produce  $u\bar{d} \rightarrow W^+$ . What you are looking for is not a virtual  $W$ , but a *real*  $W$ . The  $W$ -boson then decays into, say,  $W^+ \rightarrow e^+\nu_e$ . Of course you do not see  $\nu_e$ . The point is that the unseen neutrino carries away the energy  $m_W/2 \sim 40$  GeV. There is a lot of energy missing in the event, against the energetic electron on the other side. The problem is that we do not know the energy/momentum of the partons, namely the up-quark in the proton and the anti-down quark in the anti-proton in this case, a priori. Sure enough, partons are moving as a part of the (anti-)proton, and their momentum is a fraction of the (anti-)proton momentum, but we do not know the fraction. Then the produced  $W$  is in general moving along the beam direction. However, it is not expected to move transverse to the beam direction with a significant momentum because the partons are moving along the beam direction. Of course, the partons are confined inside the hadrons of size 0.7fm or so, and there is the uncertainty in the parton momentum of order  $\Delta p \sim \hbar/0.7\text{fm} = 300$  MeV. But relative to the momentum of order 40 GeV we are interested in, this is a small correction. Therefore, neglecting this ‘‘Fermi motion’’ of partons inside the hadrons, they are moving only along the beam direction (longitudinal direction), and so is the produced  $W$ -boson.

The four-momentum of the  $W$ -boson then is

$$p_W^\mu = m_W(\gamma, 0, 0, \gamma\beta), \quad (24)$$

where I took the  $z$ -direction to be the beam direction. In the rest frame of the  $W$ -boson, it decays into an electron and a neutrino (we ignore the electron mass in this discussion),

$$\hat{p}_W^\mu = m_W(1, 0, 0, 0), \quad (25)$$

$$\hat{p}_e^\mu = \frac{m_W}{2}(1, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \quad (26)$$

$$\hat{p}_{\nu_e}^\mu = \frac{m_W}{2}(1, -\sin\theta \cos\phi, -\sin\theta \sin\phi, -\cos\theta). \quad (27)$$

The hatted quantities are defined in the rest frame. To go back to the lab frame, we boost along the  $z$ -direction,

$$p_W^\mu = m_W(\gamma, 0, 0, \gamma\beta), \quad (28)$$

$$p_e^\mu = \frac{m_W}{2}(\gamma + \gamma\beta \cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi, \gamma \cos \theta + \gamma\beta), \quad (29)$$

$$p_{\nu_e}^\mu = \frac{m_W}{2}(\gamma - \gamma\beta \cos \theta, -\sin \theta \cos \phi, -\sin \theta \sin \phi, -\gamma \cos \theta + \gamma\beta). \quad (30)$$

The point here is that the transverse momentum of the electron and the neutrino,

$$p_T^e = \sqrt{|p_e^1|^2 + |p_e^2|^2} = \frac{m_W}{2} \sin \theta \quad (31)$$

does not depend on the boost along the beam direction. On the other hand, the  $p_T$  has a distribution rather than a definite value.

It turns out that  $p_T$  distribution is peaked at its maximum value  $m_W/2$ . The reason is in a simple phase space factor. The phase space in the  $W$ -decay  $d\Omega = d \cos \theta d\phi$  can be rewritten in terms of  $p_T$  using the relation above.  $\cos \theta = \sqrt{1 - 4p_T^2/m_W^2}$ , and hence

$$d\Omega = \frac{4p_T/m_W^2}{\sqrt{1 - 4p_T^2/m_W^2}} dp_T d\phi. \quad (32)$$

The Jacobian is singular at the maximum  $p_T = m_W/2$  and produces a peak there called ‘‘Jacobian peak.’’ Thanks to this simple kinematics, a large fraction of  $W$  events have both electron and neutrino transverse momenta close to the maximum, making the observation easier.

The trick then is to build a detector as ‘‘hermetic’’ as possible. A ‘‘hermetic’’ detector covers most of the solid angle around the collision point, so that few particles escape the detector. Basically, you don’t want any ‘‘holes.’’ In practice, you cannot place a detector along the beam axis because they get burnt too quickly. You don’t want to disturb the beam either. But you don’t care so much about having particles escaping along the beam direction, because they carry little transverse momentum. As long as you are looking for signals based on the transverse momentum, you lose little transverse momentum in your event due to the hole along the beam direction. Then you sum up all the transverse momentum you have observed as vectors in (x,y) plane, and ask how much you don’t see, *i.e.*, the ‘‘missing transverse momentum.’’ You regard this quantity to be the transverse momentum of the neutrino. Therefore, you look for an energetic electron with large transverse momentum, and also for a large missing transverse momentum. UA1 collaboration lead by Carlo Rubbia showed that looking for each of them result in

the same set of events, namely a set of events with *both* a high- $p_T$  electron and a large missing  $p_T$ . They had five events all together when they reported the discovery of the  $W$ -boson. And the largest  $p_T$  in this set of events agreed with roughly 40 GeV, consistent with the expectation. For the discoveries of  $W$  and  $Z$  boson, van der Meer and Rubbia shared the Nobel prize in 1984.

In more recent high-statistics samples gathered at Tevatron, it is more customary to look at the “transverse mass,” defined by

$$m_T = \sqrt{(p_T^e + p_T^\nu)^2 - (\vec{p}_T^e + \vec{p}_T^\nu)^2}. \quad (33)$$

This is analogous to the definition of the usual mass  $m = \sqrt{E^2 - \vec{p}^2}$ , except that only transverse quantities are used and hence is boost invariant along the beam direction. It does not necessarily assume that the  $\vec{p}_T^e + \vec{p}_T^\nu$  is strictly zero either, because it is there due to the Fermi motion of partons. Using the idealized limit again with no Fermi motion, this quantity is  $m_T = m_W \sin \theta$ , and the distribution again has the Jacobian peak. It is smeared beyond  $m_W$  due to the resolution effect. The current world average is  $m_W = 80.423 \pm 0.039$  GeV.

Just a word on nomenclature. Because the electron kinematics is measured mostly using the calorimeter, it is customary to use  $E_T = E \sin \theta$  for electrons. On the other hand, the muon kinematics is measured using the tracking detector, and we use  $p_T$ . They are of course the same for nearly massless particles such as electrons and muons, but this is the notation used in the literature.

The UA1 experiment has also discovered the  $Z$  boson using its decay  $Z \rightarrow e^+e^-$  and  $Z \rightarrow \mu^+\mu^-$ . This case, you can use the full four-momentum information because you detect both decay products with no missing momenta. A much improved data set from Tevatron is shown below.

The Fig. 5 shows lego plot of the  $W$  events. This type of plot is called lego-plot, showing the energy deposit in the calorimeter after opening the cylinder. The long direction is the beam direction, while the short one the azimuth around the beam. The beam direction is shown in terms of the pseudo-rapidity

$$\eta = \frac{1}{2} \log \frac{1 + \cos \theta}{1 - \cos \theta} = -\log \tan \frac{\theta}{2} \quad (34)$$

which is useful because it only shifts under the boost. We can see this as follows. For massless particles, the pseudo-rapidity coincides with the

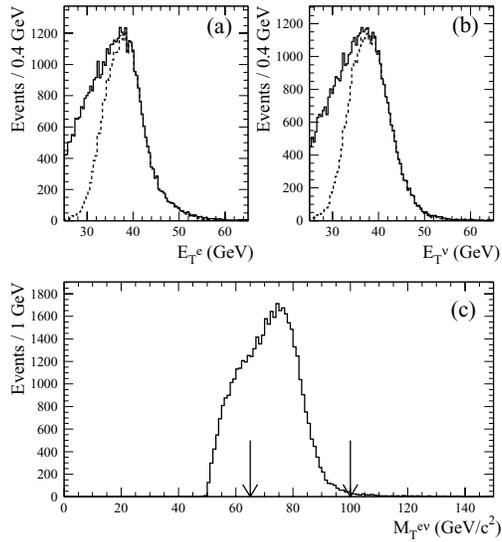


Figure 3: Kinematic quantities used in  $W \rightarrow e\nu$  sample in CDF experiment at Tevatron  $p\bar{p}$  collider.  $E_T$  distributions of (a) electron and (b) neutrinos. The dashed curves show the events in  $65 < M_T < 100$  GeV, the fit region for the  $W$  mass measurement. (c) Transverse Mass distribution. The arrows indicate the region used in the  $W$  mass fit. Taken from “Measurement of the  $W$  Boson Mass with the Collider Detector at Fermilab,” *Phys. Rev.* **D64**, 052001 (2001), <http://link.aps.org/abstract/PRD/V64/E052001/>.

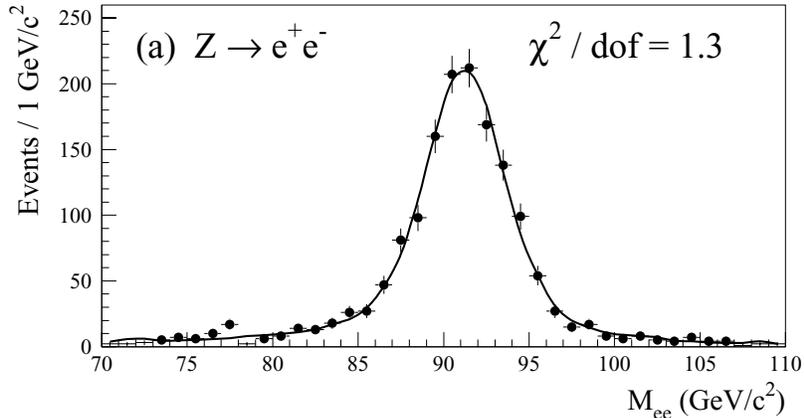


Figure 4: Invariant mass distribution. The points are the data, and the solid line is the Monte Carlo simulation (normalized to the data) with best fit. Taken from “Measurement of the  $W$  Boson Mass with the Collider Detector at Fermilab,” *Phys. Rev.* **D64**, 052001 (2001), <http://link.aps.org/abstract/PRD/V64/E052001/>.

rapidity

$$y = \frac{1}{2} \log \frac{E + p_z}{E - p_z}. \quad (35)$$

Under boost,  $E \rightarrow E\gamma + p_z\gamma\beta$ ,  $p_z \rightarrow p_z\gamma + E\gamma\beta$ . Therefore,

$$y \rightarrow \frac{1}{2} \log \frac{E(\gamma + \gamma\beta) + p_z(\gamma\beta + \gamma)}{E(\gamma - \gamma\beta) + p_z(\gamma\beta - \gamma)} = y + \frac{1}{2} \log \frac{1 + \beta}{1 - \beta}. \quad (36)$$

Therefore the separation in pseudo-rapidities among energy deposits in a single event is boost invariant. In this event, the fact that there is only one dominant energy deposit means there is a large missing transverse energy. In the similar lego plot for the  $Z$ -event, however, two energy deposits appear at azimuths different by  $180^\circ$ , and hence back-to-back in the transverse plane. There is no apparent missing transverse momentum.

## 5.2 LEP

When LEP (Large Electron Positron collider) at CERN, Geneva, started to produce millions of  $Z$ -boson, it became possible to measure the  $Z$  coupling to different particle species directly.

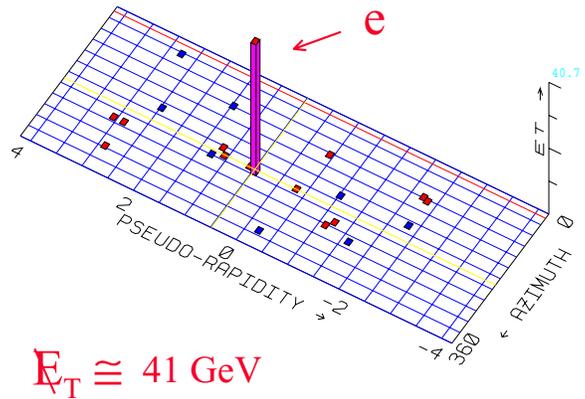


Figure 5: Event display of  $p\bar{p} \rightarrow W$  followed by  $W \rightarrow e\nu_e$  at CDF, Tevatron. The parton-level process is  $u\bar{d} \rightarrow W^+$  etc.

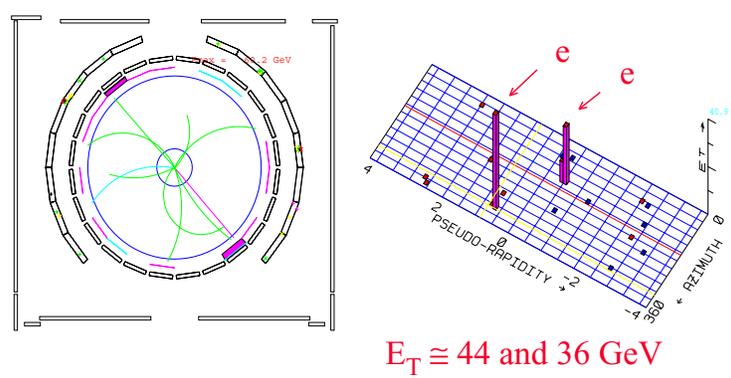


Figure 6: Event display of  $p\bar{p} \rightarrow Z$  followed by  $Z \rightarrow e^+e^-$  at CDF, Tevatron. The parton-level process is  $u\bar{u} \rightarrow Z$  etc.

What it does it to collide beams of electrons and positrons around the mass of the  $Z$ -boson. Unlike the measurement of the  $m_Z$  at  $Spp\bar{S}$  or Tevatron where the kinematics of the final state particles are measured and combined to determine the mass, here directly the beam energy is used to determine the mass. Beam energies can be measured far more accurately than the particle kinematics in a detector. In fact, the energy of the LEP beams had been calibrated to an extraordinary accuracy, that the effect of tides from the gravity of the moon on the size of the ring had been seen at 10MeV level and corrected for. Bob Jacobsen in Department had played a major role in this beam energy calibration (see *Nucl. Instrum. Meth.* **A357**, 249-252 (1995)). An unanticipated effect was also seen. They observed that there was a mysterious fluctuation in the beam energy was observed only during the day time but not from midnight to five in the morning. The NMR (Nuclear Magnetic Resonance) probe detected the variation in the magnetic field in the bending magnet of the accelerator. It turned out that the TGV, the bullet train whose railtrack runs nearby the LEP tunnel, affects the current in the ground water that in turn effected the magnet and hence the beam energy. After these heroic efforts, the mass of the  $Z$ -boson had been determined to  $m_Z = 91.1876 \pm 0.0021$ . This is an unbelievable precision.

One important quantity that had been measured by LEP is that there are only three generations of particles with (near-)massless neutrinos. Because the  $Z \rightarrow \nu\bar{\nu}$  decay is possible, even though you do not see this in your detector, it affects the width of the  $Z$  resonance. More neutrinos would make the  $Z$ -boson wider. Note that a wider resonance makes the peak lower. The data clearly exclude more than three neutrinos.

At the peak of the  $Z$ -resonance, we are looking at the process  $e^+e^- \rightarrow Z \rightarrow f\bar{f}$  with little interference with the photon exchange. The final state fermion can be  $f = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b$  with both helicities (except for the purely left-handed neutrinos). There are many important measurements that can be understood easily.

Take  $e^+e^- \rightarrow Z \rightarrow \mu^+\mu^-$ . Because of the angular momentum consideration we used in determining the spin of the quark before, the four angular distributions are proportional to

$$\frac{d\sigma}{d\cos\theta}(e_L^-e_R^+ \rightarrow \mu_L^-\mu_R^+) \propto |g_L^e|^2|g_L^\mu|^2(1 + \cos\theta)^2, \quad (37)$$

$$\frac{d\sigma}{d\cos\theta}(e_L^-e_R^+ \rightarrow \mu_R^-\mu_L^+) \propto |g_L^e|^2|g_R^\mu|^2(1 - \cos\theta)^2, \quad (38)$$

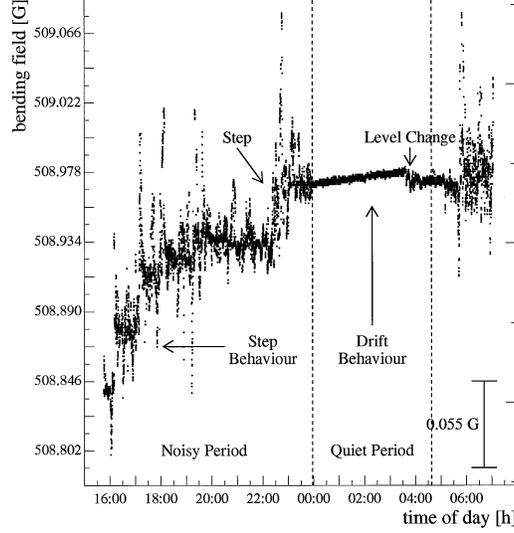


Figure 7: Magnetic field evolution measured in the tunnel by NMR8. The field increase during this period shows variations of the slope and steps of various sizes. Taken from *Nucl. Instrum. Meth.* **A417**, 9-15 (1998).

$$\frac{d\sigma}{d\cos\theta}(e^-e^+ \rightarrow \mu^- \mu^+) \propto |g_R^e|^2 |g_L^\mu|^2 (1 - \cos\theta)^2, \quad (39)$$

$$\frac{d\sigma}{d\cos\theta}(e^-e^+ \rightarrow \mu^- \mu^+) \propto |g_R^e|^2 |g_R^\mu|^2 (1 + \cos\theta)^2. \quad (40)$$

Here,  $g_L^e = g_L^\mu = (I_3^e - Q_e \sin^2 \theta_W) = -\frac{1}{2} + \sin^2 \theta_W$ ,  $g_R^e = g_R^\mu = -Q_e \sin^2 \theta_W = \sin^2 \theta_W$  are the couplings of the particle species to the  $Z$ -boson (with the overall size  $g_Z$  stripped because it is common to all processes). If you look at the total “forward” events, namely  $\cos\theta > 1$ , the total will be given by

$$\begin{aligned} & \int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta}(e^-e^+ \rightarrow \mu^- \mu^+) \\ & \propto \frac{7}{3}(|g_L^e|^2 |g_L^\mu|^2 + |g_R^e|^2 |g_R^\mu|^2) + \frac{1}{3}(|g_L^e|^2 |g_R^\mu|^2 + |g_R^e|^2 |g_L^\mu|^2), \end{aligned} \quad (41)$$

and the backward events

$$\begin{aligned} & \int_{-1}^- d\cos\theta \frac{d\sigma}{d\cos\theta}(e^-e^+ \rightarrow \mu^- \mu^+) \\ & \propto \frac{1}{3}(|g_L^e|^2 |g_L^\mu|^2 + |g_R^e|^2 |g_R^\mu|^2) + \frac{7}{3}(|g_L^e|^2 |g_R^\mu|^2 + |g_R^e|^2 |g_L^\mu|^2). \end{aligned} \quad (42)$$

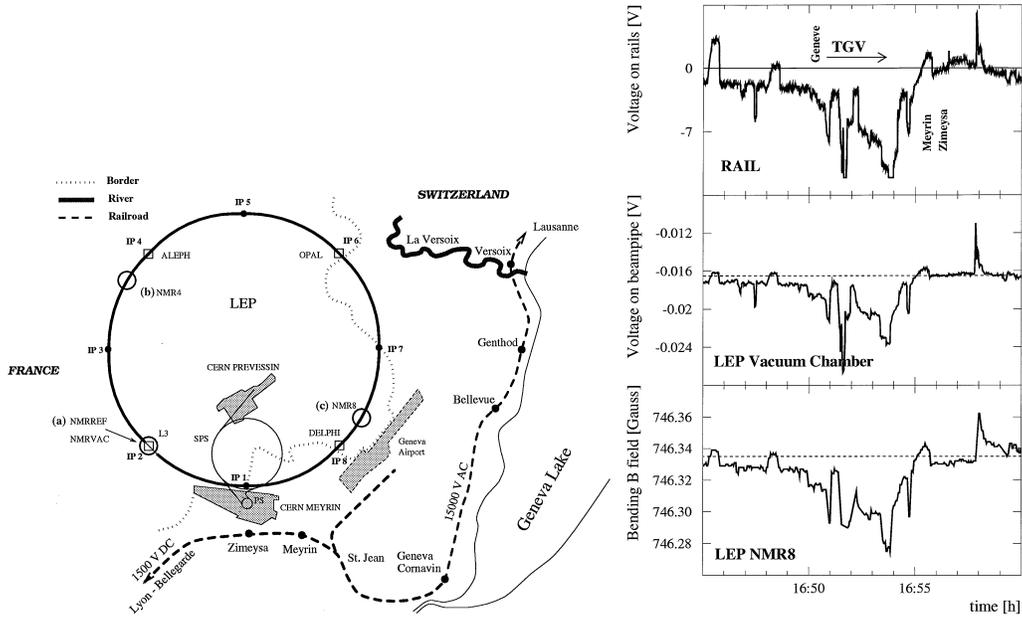


Figure 8: LEFT: The LEP ring surrounded by the French and Swiss railroads with the locations of the NMR probes and the four experiments. Two probes are installed in a reference dipole magnet (a), which is connected in series with the LEP dipoles. NMR4 (b) and NMR8 (c) are mounted directly in LEP dipoles in the tunnel. RIGHT: Train leakage currents, vacuum chamber currents and the associated magnetic field perturbation on Nov. 13th, 1995. The observed peaks are coincident with the departure of the 16:50 Geneva–Paris TGV (SNCF). Taken from *Nucl. Instrum. Meth.* **A417**, 9-15 (1998).

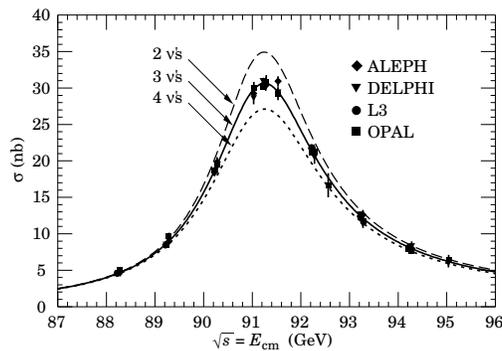


Figure 9: The lineshape of the Z-boson. Data vs calculations for various numbers of neutrinos.

This allows us to define the forward-backward asymmetry,

$$\begin{aligned}
A_{FB}^\mu &= \frac{\sigma_F(\mu) - \sigma_B(\mu)}{\sigma_F(\mu) + \sigma_B(\mu)} \\
&= \frac{3(|g_L^e|^2|g_L^\mu|^2 + |g_R^e|^2|g_R^\mu|^2) - (|g_L^e|^2|g_R^\mu|^2 + |g_R^e|^2|g_L^\mu|^2)}{4(|g_L^e|^2|g_L^\mu|^2 + |g_R^e|^2|g_R^\mu|^2)(|g_L^e|^2|g_R^\mu|^2 + |g_R^e|^2|g_L^\mu|^2)} \\
&= \frac{3(|g_L^e|^2 - |g_R^e|^2)(|g_L^\mu|^2 - |g_R^\mu|^2)}{4(|g_L^e|^2 + |g_R^e|^2)(|g_L^\mu|^2 + |g_R^\mu|^2)}.
\end{aligned} \tag{43}$$

This is a simple counting experiment with very little systematic problems, and can be measured limited only by statistics. And the statistics is large at the peak of the resonance. This quantity is one of the best way to measure  $\sin^2 \theta_W$ . In fact, this measurement can be done for different particle species one by one, and one can compare different measurements of  $\sin^2 \theta_W$ , the test of neutral-current universality.

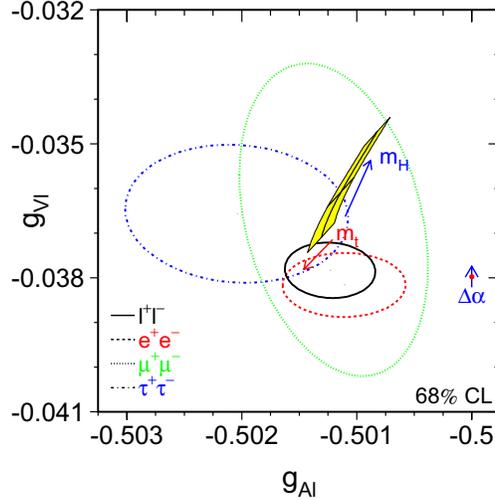


Figure 10: The universality test of charged lepton couplings to the  $Z$ -boson from LEP. The axial and vector couplings are defined using the left-handed and right-handed couplings,  $g_A = g_R - g_L$ ,  $g_V = g_R + g_L$ .

More recently LEP had been upgraded to run above the threshold for  $e^+e^- \rightarrow W^+W^-$ . The  $W^+$ -boson can decay into  $u\bar{d}'$ ,  $c\bar{s}'$ ,  $e^+\nu_e$ ,  $\mu^+\nu_\mu$ ,  $\tau^+\nu_\tau$ . Because the quark final states come with three colors, but all final states are universal, they are equally divided up to one part in 9. Hence the branching

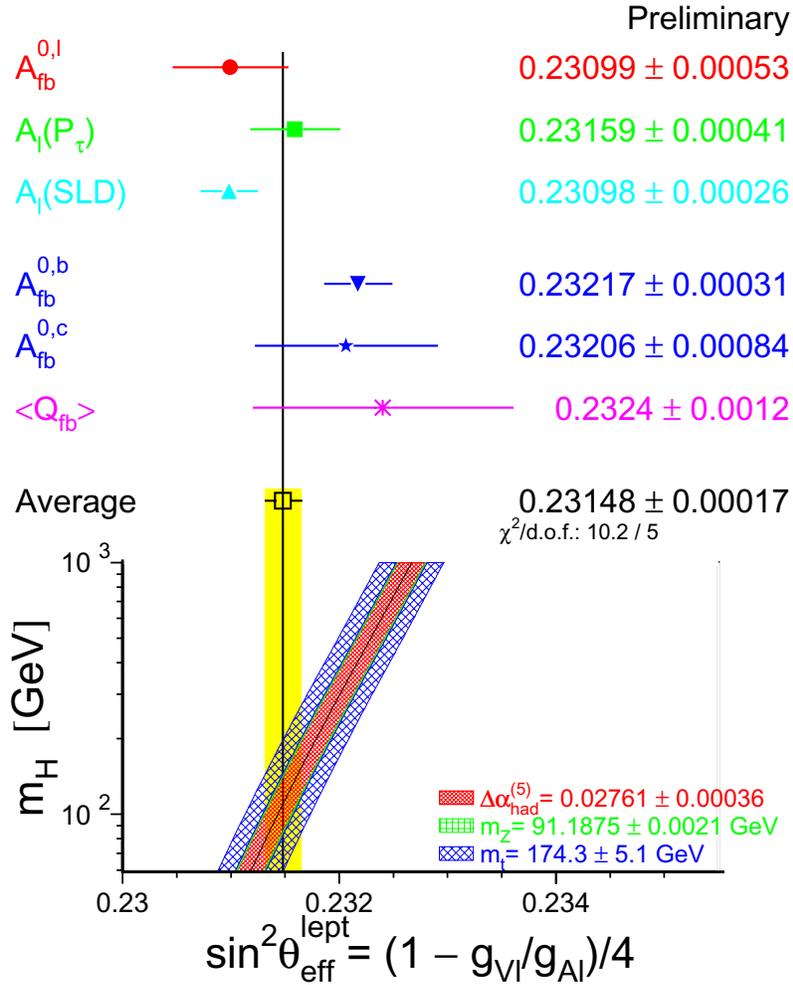


Figure 11: Comparison of different measurements of  $\sin^2 \theta_W$ . The spread is a little bit more than usual, but nonetheless they agree at a very high accuracy. Taken from LEP Electroweak Working Group <http://www.cern.ch/LEPEWWG>.

fraction to each leptonic state is about  $1/9$ . There is a small enhancement due to additional gluon emission at the level of  $\alpha_s/\pi \simeq 4\%$  for quark modes. This makes the branching fractions to be

$$BR(W \rightarrow \text{hadrons}) = \frac{6(1 + \alpha_s/\pi)}{6(1 + \alpha_s/\pi) + 3}, \quad (44)$$

$$BR(W \rightarrow e\nu_e) = BR(W \rightarrow \mu\nu_\mu) = BR(W \rightarrow \tau\nu_\tau) = \frac{1}{6(1 + \alpha_s/\pi) + 3} \quad (45)$$

The data are completely consistent with the expectation.

$e\nu_e$	$10.72 \pm 0.16\%$
$\mu\nu_\mu$	$10.57 \pm 0.22\%$
$\tau\nu_\tau$	$10.74 \pm 0.27\%$
hadrons	$67.96 \pm 0.35\%$

Table 1: Branching fractions of  $W$  into various final states. Taken from PDG 2002.

Look at <http://alephwww.cern.ch/DALI/192GeV/> for  $W$ -pair events with various decay patterns. The production cross section of  $W$ -pair rises quickly above the threshold, and is again consistent with the expectation.

## 6 Top Quark

Since the tau lepton was found in 1975, it was expected that it is accompanied by the quarks as the previous two generations. Indeed the bottom quark was found in the Upsilon resonances in 1978, and the search for the top quark started. It was not until 1995 that it was finally discovered. It was much heavier than anybody anticipated. Compared to the tau lepton (1.777 GeV) and the bottom quark (about 4–5 GeV depending on how you define the mass of a quark), the top quark weighs whopping  $174.3 \pm 5.1$  GeV. Because it is so heavy, it can decay into a *real*  $W$  instead of a virtual one,  $t \rightarrow bW^+$ . At Tevatron, the dominant production mechanism is  $u\bar{u} \rightarrow g^* \rightarrow t\bar{t}$ , where  $g^*$  is the virtual gluon. Then both  $t$  and  $\bar{t}$  decay quickly into  $bW^+$  and  $\bar{b}W^-$  before the top quarks realize they have to hadronize. The  $W$ -bosons further decay into jets or lepton-neutrino pairs. The search at Tevatron relies on the lepton final states.

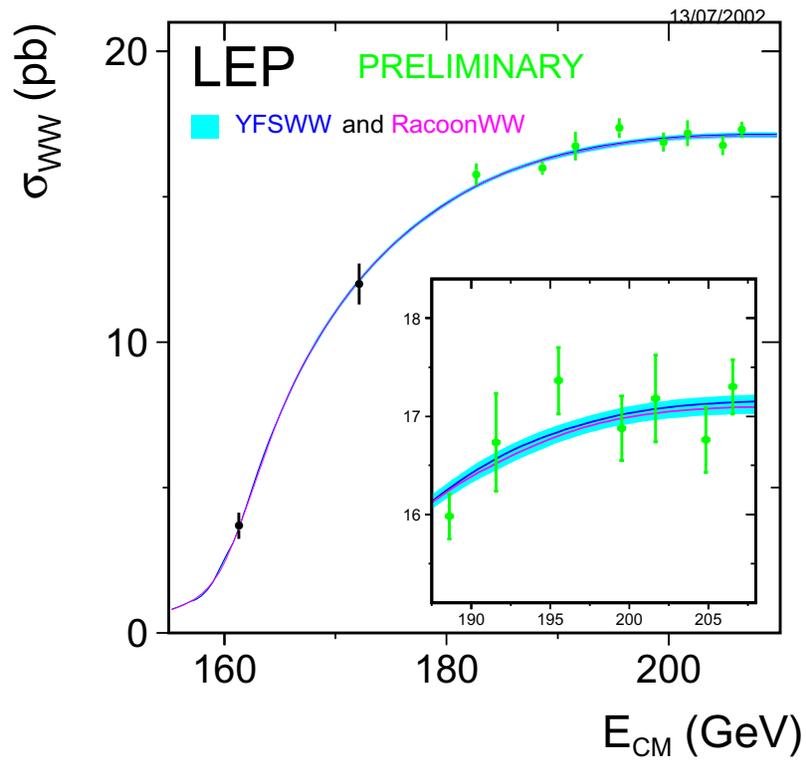


Figure 12: The measured and predicted cross section for  $e^+e^- \rightarrow W^+W^-$ . Taken from <http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer02/>.

# e + 4 jet event

40758\_44414

24-September, 1992

**TWO jets tagged by SVX**

**fit top mass is 170 +/- 10 GeV**

$e^+$ , Missing  $E_T$ , jet #4 from top  
jets 1,2,3 from top ( 2&3 from W )

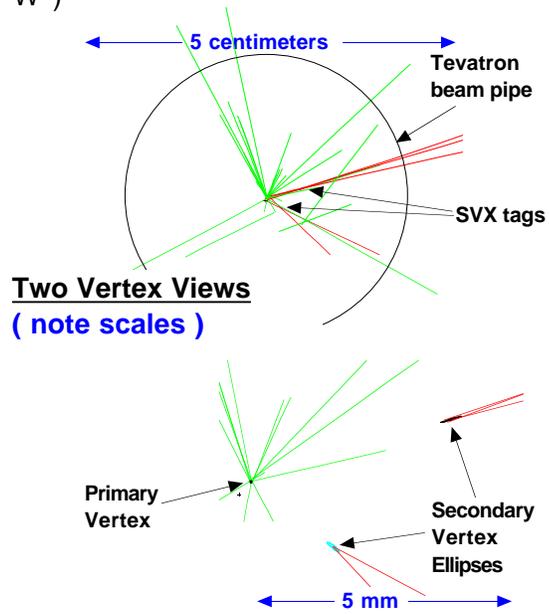
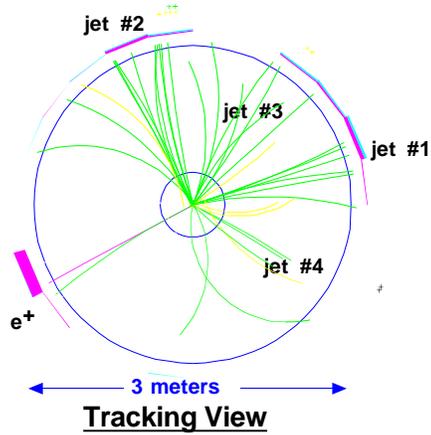
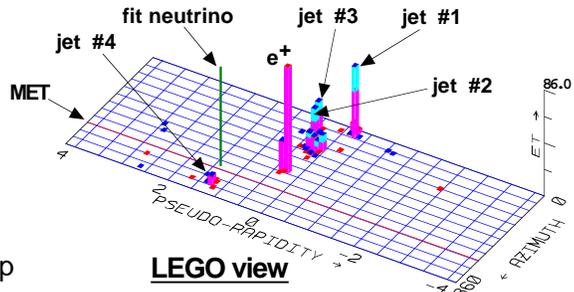


Figure 13: One of the very first candidate events for the top quark at CDF, Tevatron. It shows two jets of the bottom quarks with long-lived tracks, one positron and one neutrino (missing transverse momentum), and two additional jets.

## 7 Kobayashi–Maskawa Theory

Now that there are three generations of quarks, clearly the Cabibbo angle must be generalized to include full three generations. It turns out that it is not just going from a two-by-two rotation to a three-by-three rotation. It is essential to our understanding of CP violation.

The left-handed quarks come in weak isospin doublets,

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \quad \begin{pmatrix} c \\ s' \end{pmatrix}, \quad \begin{pmatrix} t \\ b' \end{pmatrix}. \quad (46)$$

The story is the same as Cabibbo's to the extent that  $d'$ ,  $s'$ , and  $b'$  are linear combinations of the mass eigenstates  $d$ ,  $s$ , and  $b$ . The difference is that we allow arbitrary *complex* coefficients keeping the state normalized, and therefore we deal with a *unitarity* matrix instead of a *rotation* matrix. Namely,

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad (47)$$

where

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad (48)$$

is the Cabibbo–Kobayashi–Maskawa matrix. All elements are in general complex, subject to the usual constraints of unitarity  $V_{CKM}^\dagger V_{CKM} = V_{CKM} V_{CKM}^\dagger = I$ .

The important point by Kobayashi–Maskawa back in 1973 (*Prog. Theor. Phys.* **49**, 652-657 (1973)) is that not all phases are physical. Because the overall phase of a state cannot be observed, we are free to choose the phase of, in our case,  $d$ ,  $s$ ,  $b$ , and also  $d'$ ,  $s'$ ,  $b'$  states. That allows us to eliminate most of the phases present in the CKM matrix. A three-by-three unitarity matrix has nine parameters. It appears that we can change the phases of six states to remove six parameters so that we are left with only three, exactly the number of parameters of a three-by-three rotation matrix. However, an overall phase rotation of all six quark states by the same phase actually does not change  $V_{CKM}$ . Therefore an effective number of phase rotation you can do to remove unphysical degrees of freedom in the  $V_{CKM}$  is five instead of

six, and you are left with four parameters: three angles for a rotation matrix and one phase. Therefore this remaining phase is physical. This is the key to understand CP violation.

In general, if you start with a  $N \times N$  unitarity matrix, it has  $N^2$  parameters. Following the same argument you can choose the phase of  $2N$  quark states to remove phases from the matrix elements, but one overall phase does not do anything. This way, we can eliminate  $2N - 1$  unphysical parameters, with  $N^2 - (2N - 1) = N^2 - 2N + 1$  parameters left. Among them,  $N(N - 1)/2$  are parameters of a rotation matrix, while  $(N^2 - 2N + 1) - N(N - 1)/2 = (N - 1)(N - 2)/2$  phases remain physical. For the two-generation case of Cabibbo, there is one angle and no phase. Therefore, what Cabibbo considered was actually completely general. Even if he had allowed complex matrix elements, he would have ended up with the same proposal. However with three generations and more, there always remain physical phases. This is what Kobayashi and Maskawa pointed out to explain the observed CP violation in the  $K^0$ - $\bar{K}^0$  system. It is remarkable that they dared to propose this as an explanation a year before charm was discovered.

The most general form of the CKM matrix is given by

$$\begin{aligned}
 V_{CKM} &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\
 &= \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}. \quad (49)
 \end{aligned}$$

The notation is  $c_{ij} = \cos \theta_{ij}$ ,  $s_{ij} = \sin \theta_{ij}$  for three angles  $\theta_{12}$ ,  $\theta_{13}$ ,  $\theta_{23}$ , and  $\delta$  is the phase that violates CP. The last expression is analagous to the use of Euler angles to write down any rotation matrix in three dimensions, except that the middle matrix has a phase. It is easy to check that this parameterization always makes the CKM matrix unitarity with this product form. In the limit  $\theta_{13} = \theta_{23} = 0$ , it reduces to the Cabibbo mixing. As long as  $\theta_{13} \neq 0$ , the phase  $\delta$  is physical. Empirically, the CKM matrix is well approximated in the so-called Wolfenstein parameterization which is (conceptually) a Taylor expansion in the powers of the Cabibbo angle

$\lambda \equiv \sin \theta_C = s_{12} \approx 0.22$ .

$$V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4). \quad (50)$$

This expression is meant to give the approximate size of matrix elements with  $\lambda \approx 0.22$  being the only small parameter. Other parameters,  $A$ ,  $\rho$ ,  $\eta$  are all supposed to be  $O(1)$ . Indeed,  $A \approx 0.8$  and  $\rho$ ,  $\eta$  are also  $O(1)$  (actually about 0.2 and 0.4) as shown in Fig. 15.

Note that the element  $V_{cb} \approx A\lambda^2 \ll \lambda$ . This makes the decay of the bottom quark  $b \rightarrow c\ell^- \nu_\ell$  suppressed by  $|V_{cb}|^2$ , and the lifetime is unexpectedly long. Thanks to this fact, we can measure decay vertex of  $B$ -mesons to study CP violation.

How does the phase in the CKM matrix help us understand the CP violation? In the case of neutral kaon, we had discussed that the so-called box diagram induces the mixing, going through up- and charm-quark intermediate states. With the full three-generation CKM matrix, the top quark comes in as well. By drawing the diagram, it is easy to see that the top quark contribution to  $M_{12}$  (the off-diagonal element in the neutral kaon Hamiltonian) comes with the CKM factor  $(V_{td}V_{ts}^*)^2$ . Using Wolfenstein parameterization, it is  $(V_{td}V_{ts}^*)^2(A\lambda^3(\rho - i\eta))^2(-A\lambda^2)^2 = A^4\lambda^{10}(\rho - i\eta)^2$ . The point is that the imaginary part of  $M_{12}$  leads to the (indirect) CP violation. This is how the CP violation arises in the neutral kaon system in Kobayashi–Maskawa theory.

In the case of  $B^0\text{--}\bar{B}^0$  mixing the same box diagram comes with the CKM factor  $(V_{td}V_{tb}^*)^2 = A^2\lambda^6(\rho - i\eta)^2$ , and again its imaginary part gives the indirect CP violation in  $B_d(\bar{B}^0) \rightarrow J/\psi K_S$ .

People often talk about “unitarity triangle(s).” This is something that works for three generations. If you write out the unitarity constraint  $V_{CKM}^\dagger V_{CKM} = I$  and take the  $db$  element of it, you find

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \quad (51)$$

This is a relationship among three complex numbers that they add up to zero. You can represent a complex number as a vector on a complex plane, and three vectors add up to zero: they form a triangle. The CP violating phase  $\Im m(V_{td}V_{tb}^*)^2$  is proportional to  $\sin 2\beta$ , and hence the result is often quoted in terms of  $\sin 2\beta$ .<sup>3</sup>

<sup>3</sup>Belle experiment uses an alternative notation,  $\sin 2\phi_1$ .

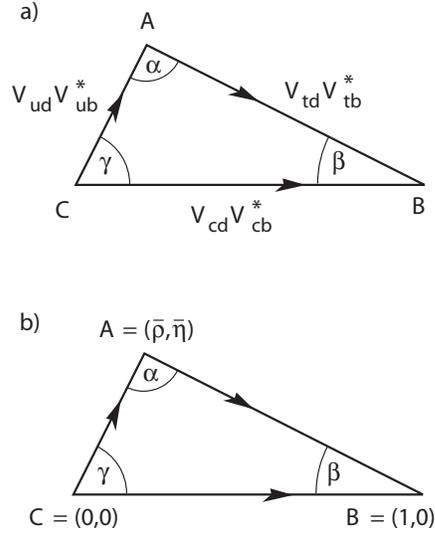


Figure 14: The unitarity triangle relevant for  $B_d$  meson system.

The recent discoveries of CP violation in the  $B_d$  meson system  $\sin 2\beta \neq 0$  was an important test of this theory. Current data tell us that all measurements of CKM matrix elements,  $|V_{ub}|$  from  $b \rightarrow u$  decays,  $|V_{td}|$  from the magnitude of  $B_d$ - $\bar{B}_d$  mixing, CP-violating parameter  $\epsilon_K$  in neutral kaons, and  $\sin 2\beta$  are all consistent for  $\rho \sim 0.2$ ,  $\eta \sim 0.4$ . This is a great success of this theory.

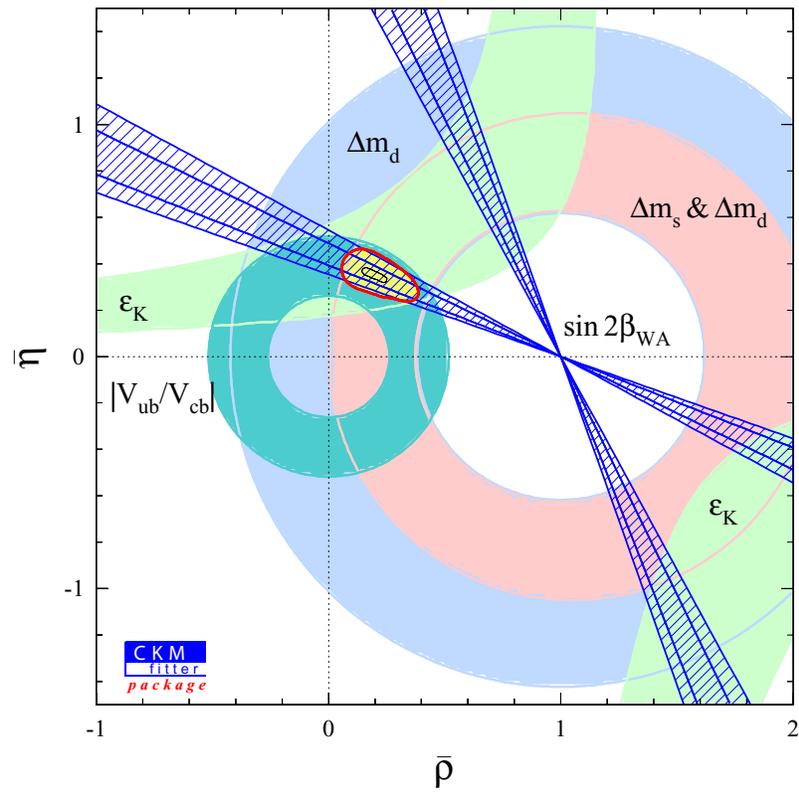


Figure 15: The recent fit to CKM parameters  $\rho$  and  $\eta$ . Taken from the talk by Yossi Nir at ICHEP2002 conference in Amsterdam.