# HW #5 (221A), due Sep 29, 4pm

1. Answer the following questions on the harmonic oscillator $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$.

   (a) Verify that the Hamiltonian can be recast to the form $H = \hbar\omega(N + \frac{1}{2})$, where $N = a^\dagger a$ and $a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P}{m\omega}\right)$.

   (b) Write down the condition for the ground state $a|0\rangle = 0$ in the position representation, solve it analytically, and normalize it properly. Plot the shape of the wave function.

   (c) Work out the wave functions of the first- and second-excited states $\langle x|1\rangle$ and $\langle x|2\rangle$ using the ground state wave function and the creation operator. Pay attention to the normalization. Plot the shapes of the wave functions.

   (d) Without using the explicit forms of the wave function, calculate $\langle X\rangle$, $\langle(\Delta X)^2\rangle$, $\langle P\rangle$, and $\langle(\Delta P)^2\rangle$ for the state $|n\rangle$.

   (e) Verify that the coherent state

   $$|f\rangle = e^{fa^\dagger}|0\rangle e^{-|f|^2/2} \tag{1}$$

   is a normalized eigenstate of the annihilation operator $a|f\rangle = f|f\rangle$. Note that $f$ is in general a *complex* number.

   (f) Calculate $\langle X\rangle$, $\langle P\rangle$, $(\Delta X)^2$, and $(\Delta P)^2$ for the coherent state, and verify that it is a minimum uncertainty state (*i.e.*, $(\Delta X)(\Delta P) = \hbar/2$).

   (g) (optional) Show that

   $$\langle x|f\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\left(\sqrt{\frac{m\omega}{2\hbar}}\,x - f\right)^2 + \frac{1}{2}(f^2 - |f|^2)\right) \tag{2}$$

   is a solution to the equation $a|f\rangle = f|f\rangle$ in the similar way as in part (b).

   (h) Using the solution to the Heisenberg equation of motion (Eq. (2.3.45) in Sakurai), calculate the expectation value of the position $\langle f|X(t)|f\rangle$ and the momentum $\langle f|P(t)|f\rangle$ in the Heisenberg picture. It is precisely what is expected in classical mechanics.

(i) (optional) To see the shape of the wave function, we'd rather use the Schrödinger picture. Show that $|f,t\rangle = U(t)|f\rangle = |fe^{-i\omega t}\rangle e^{-i\omega t/2}$.

(j) Using the wave function Eq. (2), and taking $f = \sqrt{\frac{m\omega}{2\hbar}}\, x_0 e^{-i\omega t}$, plot the probability densities at constant time intervals so that you can observe the motion of the wave. (If you use Mathematica, you can watch them in animation.)

2. (optional) For the Gaussian wave packet

$$\psi(x) = \langle x|\psi\rangle = N e^{ipx/\hbar} e^{-(x-x_0)^2/4d^2}, \tag{3}$$

you have seen in the previous homework that $\langle\psi|X|\psi\rangle = x_0$, $\langle\psi|P|\psi\rangle = p$, $\langle\psi|(\Delta X)^2|\psi\rangle = d^2$, $\langle\psi|(\Delta P)^2|\psi\rangle = \hbar^2/4d^2$. We now consider its time evolution. The Hamiltonian is that of a free particle, $H = P^2/2m$. Using the momentum space wave function from the previous homework $\phi(q) = \langle q|\psi\rangle$, the state is expanded as a linear combination of the momentum eigenstates $P|q\rangle = q|q\rangle$ as $|\psi\rangle = \int dq|q\rangle\langle q|\psi\rangle$. Therefore, the time evolution operator gives simply

$$|\psi,t\rangle = e^{-iHt/\hbar}\int dq|q\rangle\langle q|\psi\rangle = \int dq e^{-iq^2t/2m\hbar}|q\rangle\langle q|\psi\rangle \tag{4}$$

(a) Work out the wave function at a later time $\psi(x,t) = \langle x|\psi,t\rangle$.

(b) Show that the expectation value of the position moves as $\langle\psi,t|X|\psi,t\rangle = x_0 + \frac{p}{m}t$.

(c) Show that the wave packet spreads out over time by calculating $\langle\psi,t|(\Delta X)^2|\psi,t\rangle$.

(d) Write down the Heisenberg equations of motion and solve them.

(e) Using the solution to the Heisenberg equations of motion, calculate $\langle\psi|X(t)|\psi\rangle$ and $\langle\psi|(\Delta X(t))^2|\psi\rangle$ in the Heisenberg picture and compare the results to the ones obtained above in the Schrödinger picture.