Wave Packet with a Resonance

I just wanted to tell you how one can study the time evolution of the wave packet around the resonance region quite convincingly. This in my mind is the most difficult problem of the homework.

1 Separation of the Phase Shift

The main point of the analysis is that the S-matrix can be decomposed to two pieces

$$\begin{aligned} e^{2i\delta_0} &= e^{-2ika}\frac{\sin ka + \frac{\hbar^2 k}{2m\gamma}e^{ika}}{\sin ka + \frac{\hbar^2 k}{2m\gamma}e^{-ika}} \\ &= e^{-2ika} + e^{-2ika}\frac{\hbar^2 k}{2m\gamma}\frac{2i\sin ka}{\sin ka + \frac{\hbar^2 k}{2m\gamma}e^{-ika}}. \end{aligned} \tag{1}$$

The first term is that of the hard sphere scattering, and its behavior can be analyzed very easily. The second term is dominated by the pole, which we can extract.

The wave function we obtained earlier in the lecture note is

$$\psi(r) \equiv rR_0(r) = \begin{cases} \frac{\sin(ka+\delta_0)}{\sin(ka)}\sin(kr) & r < a \\ \sin(kr+\delta_0) & r > a \end{cases}. \tag{2}$$

To match with the general asymptotic form for the scattered wave $e^{ikr}e^{2i\delta_0} - e^{-ikr}$, we multiply the wave function, both inside and outside the shell, by $2ie^{i\delta_0}$ to find

$$\psi(r) = \begin{cases} \frac{e^{ika}e^{2i\delta_0}-e^{-ika}}{\sin(ka)}\sin(kr) & r < a \\ e^{ikr}e^{2i\delta_0} - e^{-ikr} & r > a \end{cases}. \tag{3}$$

The wave packet is

$$\psi = \int dq e^{-(q-k)^2 d^2}\left[e^{iqr}e^{2i\delta_0} - e^{-iqr}\right] e^{-i(\hbar q^2/2m)t}. \tag{4}$$

2 Hard Sphere Piece

Using the first term in Eq. (1), it is

$$\psi = \int dq e^{-(q-k)^2 d^2}\left[e^{iqr}e^{-2iqa} - e^{-iqr}\right] e^{-i(\hbar q^2/2m)t}. \tag{5}$$

The stationary phase approximation is valid when t is moderately large (*i.e.*, large enough that $e^{-i(\hbar q^2/2m)t}$ factor oscillates rapidly while not too large for the wave packet to start spreading). The stationary phase condition is

$$\frac{\partial}{\partial q}\left(-qr - \frac{\hbar q^2}{2m}t\right) = -r - \frac{\hbar q}{m}t = 0, \tag{6}$$

for the first term in the square bracket. This is nothing but the incoming wave packet located at $r = -(\hbar q/m)t$. Similarly the stationary phase condition for the second term in the square bracket is

$$\frac{\partial}{\partial q}\left(qr - 2qa - \frac{\hbar q^2}{2m}t\right) = r - 2a - \frac{\hbar q}{m}t = 0, \tag{7}$$

which is nothing but the scattered wave packet located at $r = (\hbar q/m)t + 2a$. Note that the term $2a$ tells us that the wave packet had not entered the shell but was bounced right by the shell.

The rest of the analysis is the same as before. We expand q around k as

$$qr - 2qa - \frac{\hbar q^2}{2m}t = kr - 2ka - \frac{\hbar k^2}{2m}t + (r - 2a - vt)(q-k) + O(q-k)^2, \tag{8}$$

ignore the second order correction, and integrate. We defined $v = \hbar k/m$ which is the velocity of the wave packet. Let us work out the behavior at large $t > 0$ for later use. Then only the first term in the square bracket in Eq. (4) contributes, and we find

$$\begin{aligned}\psi &\simeq e^{ikr}e^{-2ika}e^{-i(\hbar k^2/2m)t}\int dq e^{-(q-k)^2d^2}e^{i(r-2a-vt)(q-k)} \\ &= e^{ikr}e^{-2ika}e^{-i(\hbar k^2/2m)t}\frac{\sqrt{\pi}}{d}e^{-(r-2a-vt)^2/4d^2}.\end{aligned} \tag{9}$$

3 Resonance Piece Outside the Shell

The interest is really in the second term in Eq. (1). Therefore the piece of our interest from Eq. (4) is

$$\psi = \int dq e^{-(q-k)^2d^2}e^{iqr}e^{-2iqa}\frac{\hbar^2 q}{2m\gamma}\frac{2i\sin qa}{\sin qa + \frac{\hbar^2 q}{2m\gamma}e^{-iqa}}e^{-i(\hbar q^2/2m)t}. \tag{10}$$

First of all, it is small, suppressed by γ. Second, it has a pole. Assuming that the momentum range $q = k \pm d^{-1}$ contains only one of the resonances, but d^{-1} much wider than the width of the resonance, we can study its behavior approximately as follows.

First of all, due to the pole at the resonance close to the real axis, the integral is dominated by the region around the pole. Let us call the location of the pole to be $k_0 - i\kappa$. Since the Gaussian factor is much wider than the resonance peak, we can approximate the Gaussian factor by a constant, namely $e^{-(k-k_0)^2 d^2}$. The phase factors can also be approximated at $q = k_0 + q'$ up to the linear order in q'. Then the integral is approximately

$$\begin{aligned}\psi &= e^{-(k-k_0)^2 d^2} e^{ik_0 r} e^{-2ik_0 a} e^{-i(\hbar k_0^2/2m)t} \\ & \int dq' \frac{\hbar^2 q}{2m\gamma} e^{iq'(r-2a-vt)} \frac{2i \sin qa}{\sin qa + \frac{\hbar^2 q}{2m\gamma} e^{-iqa}}. \end{aligned} \qquad (11)$$

Because the integrand is damped strongly as q' deviates from zero, we can pretend that the integration is from $-\infty$ to ∞. The point then is that one can extend the contour to go back at the infinity on the upper or lower half plane depending on the coefficient of iq' in the exponent is positive or negative, respectively. The contour in the upper half plane does not enclose any pole, and hence the integral vanishes. Therefore the integral is non-vanishing only if the coefficient of iq' in the exponent is negative, *i.e.*, for $t > (r - 2a)/v$. The contribution is there only *after* the scattering takes place, a resonable result. The important point is that the result of the integral would be just the substitution of $q' = -i\kappa$ or $q = k_0 - i\kappa$ in the integrand (except the denominator), which is basically

$$e^{iqr} \sin qa e^{-i(\hbar q^2/2m)t} \qquad (12)$$

evaluated at $q = k_0 - i\kappa$. This is precisely the form of the wave function we obtained for the complex momentum at the pole! In other words the large t behavior of the scattered wave is nothing but the "resonance wave function" for a complex energy eigenvalue. However, there is one notable difference. The contribution exists only for $(\hbar k_0/m)t > r - a$. The exponential rise of the resonance wave function e^{iqr} becomes appreciable only for $r > \kappa^{-1}$, which requires $t > \kappa^{-1}(m/\hbar k_0) = 2\hbar/\Gamma$. In other words, by the time exponential rise starts to show, the exponential decay $e^{-\Gamma t/2\hbar}$ sets in. The exponential therefore never becomes a true enhancement. The pathological non-renormalizability of the resonance wave function is hence a non-issue.

To work out more detailed form of the wave packet, let us perform the integration explicitly. Around the pole $q = k_0 - i\kappa$, we expand the denominator as

$$\begin{aligned}\sin qa + \frac{\hbar^2 q}{2m\gamma} e^{-iqa} &= 0 + \left(\cos qa + \frac{\hbar^2}{2m\gamma} e^{-iqa} - ia \frac{\hbar^2 q}{2m\gamma} e^{-iqa}\right)(q - k_0 + i\kappa) \\ &\quad + \text{higher order.}\end{aligned} \tag{13}$$

Once we know how we pick the contribution of the pole, it is easier to go back to Eq. (10) to perform the integration. It gives

$$\psi = -2\pi i e^{-(q-k)^2 d^2} e^{iq(r-2a)} \frac{\hbar^2 q}{2m\gamma} \frac{2i \sin qa}{\cos qa + \frac{\hbar^2}{2m\gamma} e^{-iqa} - ia \frac{\hbar^2 q}{2m\gamma} e^{-iqa}} e^{-i(\hbar q^2/2m)t} \Bigg|_{q=k_0-i\kappa} . \tag{14}$$

By further using the condition for the pole, $\sin qa$ in the numerator can be rewritten as $-\frac{\hbar^2 q}{2m\gamma} e^{-iqa}$. Also, to a good approximation (large γ), $k_0 = n\pi/a + O(\gamma^{-1})$, $\kappa = O(\gamma^{-2})$ and hence $e^{-iqa} = \cos qa = (-1)^n + O(\gamma^{-1})$ and $\sin qa = O(\gamma^{-1})$. Putting them together, we find

$$\psi = -4\pi e^{-(q-k)^2 d^2} e^{iq(r-2a)} e^{-i(\hbar q^2/2m)t} \left(\frac{\hbar^2 q}{2m\gamma}\right)^2 \Bigg|_{q=k_0-i\kappa} \theta(vt - r - 2a). \tag{15}$$

We resurrected the condition $vt - r - 2a > 0$ for this contribution to exist.

4 Total Wave Packet Outside the Shell

The wave packet after the scattering is the sum of Eqs. (9) and (15) and we find

$$\begin{aligned}\psi \simeq \theta(vt - r - 2a) \Bigg[& e^{ik(r-2a)} e^{-i(\hbar k^2/2m)t} \frac{\sqrt{\pi}}{d} e^{-(r-2a-vt)^2/4d^2} \\ & -4\pi e^{-(q-k)^2 d^2} e^{iq(r-2a)} e^{-i(\hbar q^2/2m)t} \left(\frac{\hbar^2 q}{2m\gamma}\right)^2 \Bigg|_{q=k_0-i\kappa} \Bigg] .\end{aligned} \tag{16}$$

The first term is peaked around $r = 2a + vt$ with a width of d and is the bounced back wave packet by the hard sphere. The second term is the "resonance wave function" starting at $r = 2a + vt$ with an exponential tail behind due to $e^{iqr} = e^{ik_0 r + \kappa r}$ factor. Therefore the total wave packet consists of a Gaussian peak followed by an exponential tail. Another way to look at it is by placing a detector at a fixed r. Then at $t = (r+2a)/v$ you see the prompt peak followed by an exponential tail due to $|e^{-i(\hbar q^2/2m)t}|^2 = e^{-2(\hbar k_0 \kappa/m)t}$ factor. The physical interpretation of the exponential tail is precisely that of an exponentially decaying quasi-bound state. The relative phase between two terms is also important. Because we are interested in a wave packet whose momentum is peaked around the resonance $k = k_0$, the Gaussian factor $e^{-(q-k)^2 d^2}$ is basically unity, and all phase factors are the same between two terms. Therefore, there is a *destructive* interference. This is precisely what we need, because the exponential tail takes away some probability out of the bounced back wave packet, which requires the bounced back portion to be reduced to conserve probability.

5 Wave Packet Inside the Shell

The wavefunction inside the shell is

$$\psi = \int dq e^{-(q-k)^2 d^2} \frac{e^{iqa} e^{2i\delta_0} - e^{-iqa}}{\sin(qa)} \sin(qr) e^{-i(\hbar q^2/2m)t}. \tag{17}$$

Using the separation Eq. (1), the first term identically vanishes. Therefore it is only the second term we need to consider:

$$\psi = \int dq e^{-(q-k)^2 d^2} e^{-iqa} \frac{\hbar^2 q}{2m\gamma} \frac{2i}{\sin qa + \frac{\hbar^2 q}{2m\gamma} e^{-iqa}} \sin(qr) e^{-i(\hbar q^2/2m)t}. \tag{18}$$

Because the integrand is peaked highly around $q = k_0$, we can ignore the Gaussian factor as before.

The rest of the analysis is the same as the outside region. We approximate the phase factor with $q = k_0 + q'$ up to the linear order in q', and pretend that the integration is for the entire real axis. For a sufficiently large t, we can extend the contour to come back on the lower half plane, picking up the pole. Then the result is just the substitution of $q = k_0 - i\kappa$ in the integrand (except for the denominator), which is basically

$$\sin(qr) e^{-i(\hbar q^2/2m)t} \tag{19}$$

evaulated at the pole. Again, this is the "resonance wave function" we solved for a complex energy eigenvalue.

In sum, the wave packet behaves as follows. The wave packet has the velocity $v = \hbar k_0/m$. For $t < -a/v$, until the wave packet reaches the shell, it consists of only one wave packet located at $r = -vt$. For large $t > -a/v$, the wave packet consists of three pieces. One of them is just the wave packet bounced back by the shell, moving along at $r = vt + 2a$. Two other pieces are coming from the resonance, inside and outside the shell. Both of them are well described by the "resonance wave function" obtained by solving the Schrödinger equation with a complex energy eigenvalue, damping exponentially as a function of time. In other words, after the main part of the wave packet is bounced back by the shell, a small portion remains, hangs around for a while, and starts decaying only over the time scale given by the lifetime of the quasi-bound state.

6 Examples

It is interesting to also consider the case where $\Delta q = d^{-1}$ is much smaller than the width of the resonance. In that case, the wave packet is so long that the "delayed" wave packet coming from the decay of the quasi-bound state overlaps with the bounced wave packet and we cannot separate them. Such an experiment will see the Breit–Wigner shape of the resonance as a function of the energy.

An example is the production of the Z^0 boson at LEP e^+e^- collider. The energy resolution of the beam is much smaller than the width of the Z^0 boson and hence it shows the Breit–Wigner shape of the resonance as a function of the center-of-mass energy. This experiment achieved an unprecedented accuracy in high-energy physics, resulting in the determination of the mass and the width of the Z^0 boson as

$$m_Z = 91.1875 \pm 0.0021\text{GeV}, \quad (20)$$

$$\Gamma_Z = 2.4952 \pm 0.0023\text{GeV}. \quad (21)$$

The data is shown in Fig. 1 from `http://arXiv.org/abs/hep-ex/0101027`.

In the same experiment, one can see the exponential decay law of particles produced from the Z^0 decay. Fig. 2 shows an example from `http://arXiv.org/abs/hep-ex/0011083`. Here, the Z^0 boson decays into a pair of bottom quarks, and they form bound states with another quark, such as u and d

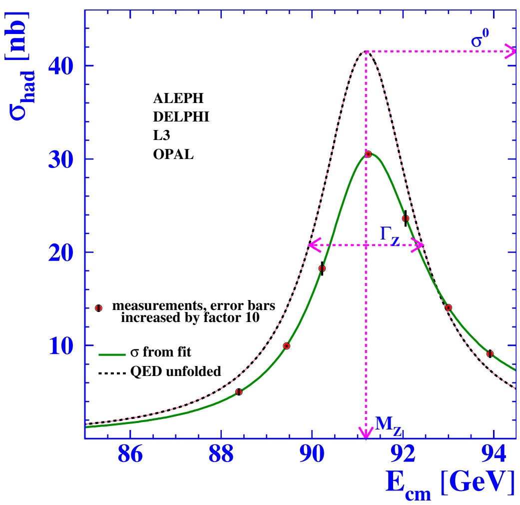

Figure 1: Average over measurements of the hadronic cross-sections by the four expeiments, as a function of centre-of-mass energy. The dashed curve shows the QED deconvoluted cross-section, which defines the Z parameters described in the paper.

due to quark confinement into mesons B^0 and B^-. These mesons are very short-lived $\tau_{\bar{\mathrm{B}}^0} = 1.518 \pm 0.053 \pm 0.034$ ps, $\tau_{\mathrm{B}^-} = 1.648 \pm 0.049 \pm 0.035$ ps. However, in the decay of the Z^0 boson, they are boosted relativistically because the energy is $E_B = m_Z c^2/2 = 45.$ GeV while their masses are about 5.279 GeV$/c^2$. The life time is hence prolonged by time dilation effect by $\gamma \sim 45.59/5.279 = 8.636$ (in practice this number is somewhat smaller because a b-quark jet produces other particles than the B-meson and they share energies). Furthermore, they run with a speed very close to the speed of light, and go for a distance of $l = c\tau\beta\gamma \simeq 0.415$ cm! This is a detectable distance using modern solid state detectors. The mesons decay into lighter mesons, in this case D-mesons. What you detect is further decay products of D-mesons. Extrapolating the tracks of the decay products back towards the e^+e^- annihilation point, one can determine the distance from the production point and the decay point of the B-mesons. To determine the proper times distance, however, you need to measure the actual energy of the B-meson *and* the decay length at the same time.

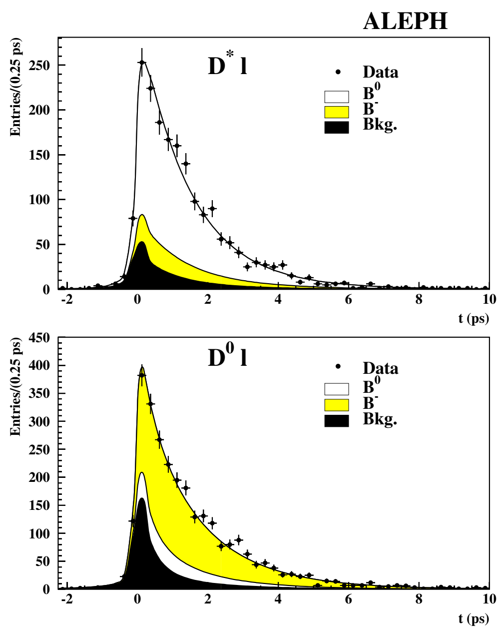

Figure 2: Measurement of B^0 and B^- meson lifetimes from $e^+e^- \rightarrow Z^0 \rightarrow b\bar{b}$, followed by formation of $B^{0,-}$ mesons due to quark confinement, and subsequent decay of the mesons. The decay of the mesons are detected by the modes $D^{*+}\ell^-$ and $D^0\ell^-$ seen in the detector away from the decay region and extrapolated back to the decay region. What are shown are proper time distributions for the $D^{*+}\ell^-$ and $D^0\ell^-$ samples, with the results of the fit superimposed. Also shown are the background contributions and the respective $\bar{B}^0$ and B^- components.