Physics 221B: Solutions to HW 1

1

Starting with the Lippman-Schwinger equation in the notes

$$|\psi>=|\phi>+\frac{1}{E-H_0+i\epsilon}V|\psi>$$

and sandwiching with $<x|$ we get

$$<x|\psi>=<x|\phi>+<x|\frac{1}{E-H_0+i\epsilon}V|\psi>\ .$$

$<x|\phi>$ is just the plane-wave wave function $\frac{1}{\sqrt{2\pi\hbar}}e^{ikx}$. The second term is

$$\begin{aligned} <x|\frac{1}{E-H_0+i\epsilon}V|\psi> &= \int dx' <x|\frac{1}{E-H_0+i\epsilon}|x'><x'|V|\psi> \\ &= \int dx' <x|\frac{1}{E-H_0+i\epsilon}|x'>V(x')\psi(x') \end{aligned}$$

where we've used the fact that V is diagonal in position space.

$$<x|\frac{1}{E-H_0+i\epsilon}|x'>=\int dp <x|\frac{1}{E-H_0+i\epsilon}|p><p|x'>=$$
$$\int dp <x|p>\frac{1}{E-\frac{p^2}{2m}+i\epsilon}<p|x'>=\int dp\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\frac{1}{E-\frac{p^2}{2m}+i\epsilon}\frac{e^{-ipx'/\hbar}}{\sqrt{2\pi\hbar}}=$$
$$\frac{2m}{2\pi\hbar}\int dp\frac{e^{ip(x-x')/\hbar}}{2mE-p^2+i\epsilon}=\frac{-m}{\pi\hbar}\int dp\frac{e^{ip(x-x')/\hbar}}{(p-\hbar k-i\epsilon)(p+\hbar k+i\epsilon)}.$$

In the last line we've used $\hbar k=\sqrt{2mE}$ and noted we can drop $O(\epsilon^2)$. (Also, note that we can multiply ϵ by any positive number without changing anything since it going to zero in the end anyway).

We will solve this integral using contour methods. Assume p is a complex variable. We want to integrate on a contour going from $-\infty$ to $+\infty$ along the real p axis. We can close the contour on an infinitely large semi-circle going either form above or bellow. We will choose the contour along which the integrand is exponentially suppressed rather than enhanced. Note that the integrand has two poles at $p=\pm(\hbar k+i\epsilon)$.

- For $x-x'>0$: We will choose the contour in the upper half-plane and pick up the pole at $p=+(k+i\epsilon)$. The integral along the closed contour (and therefore also along the real line) is $2\pi i$ times the residue:

$$\int dp\frac{e^{ip(x-x')/\hbar}}{(p-k-i\epsilon)(p+k+i\epsilon)}=$$

$$= 2\pi i(p-k-i\epsilon)\frac{e^{ip(x-x')/\hbar}}{(p-\hbar k-i\epsilon)(p+\hbar k+i\epsilon)}\Bigg|_{p=\hbar k} = 2\pi i\frac{e^{ik(x-x')}}{2\hbar k}$$

where we've taken the $\epsilon \to 0$ limit.

- For $x - x' < 0$: We will choose the contour in the lower half-plane, picking up the pole at $p = -\hbar k - i\epsilon$. then we get:

$$\int dp \frac{e^{ip(x-x')/\hbar}}{(p-\hbar k-i\epsilon)(p+\hbar k+i\epsilon)} =$$

$$= -2\pi i(p+\hbar k+i\epsilon)\frac{e^{ip(x-x')/\hbar}}{(p-\hbar k-i\epsilon)(p+\hbar k+i\epsilon)}\Bigg|_{p=-\hbar k} = 2\pi i\frac{e^{-ik(x-x')}}{2\hbar k}$$

where we've again taken $\epsilon \to 0$. The minus sign in the second step comes from integrating along the contour in the clockwise direction.

So we got

$$\int dp \frac{e^{ip(x-x')/\hbar}}{(p-\hbar k-i\epsilon)(p+\hbar k+i\epsilon)} = \begin{cases} \pi i\frac{e^{ik(x-x')}}{\hbar k} & \text{if } x-x'>0 \\ \pi i\frac{e^{-ik(x-x')}}{\hbar k} & \text{if } x-x'<0 \end{cases}$$

$$= \pi i\frac{e^{ik|x-x'|}}{\hbar k}.$$

Putting everything back together we can write the Lippman-Schwinger equation in 1D:

$$\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-mi}{\hbar^2 k}\int dx' e^{ik|x-x'|}V(x')\psi(x') \tag{1}$$

2

If the distance from the target to our detector is much larger than the spatial size of the potential a we can expand $|x-x'| = \sqrt{x^2+x'^2-2xx'} \simeq |x|(1-\frac{xx'}{|x|^2}) = r - \frac{xx'}{r}$ for $r = |x|$. Then we can rewrite eq. (1)

$$\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-mi}{\hbar^2 k}\int dx' e^{ik(r-\frac{xx'}{r})}V(x')\psi(x')$$

Defining $k' = k\frac{x}{r} = \pm k$ we get

$$\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-mi}{\hbar^2 k}e^{ikr}\int dx' e^{-ik'x'}V(x')\psi(x') = \frac{1}{\sqrt{2\pi\hbar}}\left[e^{ikx} + f(k,k')e^{ikr}\right]$$

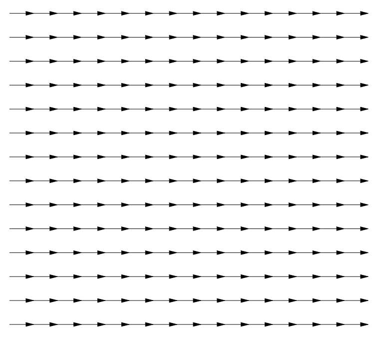
where $f(k,k') = \sqrt{2\pi\hbar}\frac{-mi}{\hbar^2 k}\int dx' e^{-ik'x'}V(x')\psi(x') = \frac{-2\pi mi}{\hbar k} < \hbar k|V|\psi >$.

Lets analyze this for both signs of x. The plane wave part is there both for $x > 0$ and $x < 0$, moving in the positive direction.The scattered wave, however, moves away from the origin in both regions. If $x > 0$ and $k' = k$ we get a scattered wave $\sim f(k,k)e^{ikx}$. For $x < 0$ and $k = -k'$ we get $f(k,-k)e^{-ikx}$.

3

The probability current is $\vec{j} = \frac{\hbar}{m}\mathrm{Im}(\psi^* \vec{\nabla}\psi)$.

1. $\psi(x) = e^{i\vec{k}\cdot\vec{x}} \implies \vec{\nabla}\psi = i\vec{k}e^{i\vec{k}\cdot\vec{x}} \implies \psi^*\vec{\nabla}\psi = i\vec{k} \implies \vec{j} = \frac{\hbar k}{m}$ which is the plane-wave's 'velocity'. $\vec{\nabla}\cdot\vec{j} = 0$, obviously. Plotting the flux with Mathematica:



Flux of plane-wave.

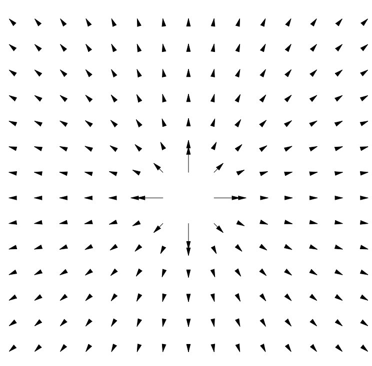
2. Using spherical coordinates, the gradient of a spherically symmetric is just $\left(\frac{\partial}{\partial r}\psi(r), 0, 0\right)$. Therefore

$$\psi(r) = \frac{e^{ikr}}{r} \implies \vec{\nabla}\psi = ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \implies \psi^*\vec{\nabla}\psi = ik\frac{1}{r^2} - \frac{1}{r^3}$$

and we get

$$\vec{j} = \frac{\hbar k}{mr^2}\hat{r},$$

which looks like this:



Flux of a spherical wave.

Lets look at $\vec{\nabla}\cdot\vec{j}$. We can use the well known result from electromagnetism $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\delta(r)$ which is indeed zero everywhere except for $r = 0$, but lets prove it. In our case $\vec{\nabla}\cdot\vec{j} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\hbar k}{mr^2}\right) = 0$ for $r \neq 0$.But, by looking at the electric field flux of a point charge at the origin

$$\int_A \frac{\hat{r}}{r^2}\cdot d\vec{a} = 4\pi$$

and using Gauss's theorem $\int_A \vec{j}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{j}dV$ we find that the divergence of $\vec{j}$ must be non-zero *somewhere*. Since we've shown that its zero everywhere but the origin, it must be

$$\vec{\nabla}\cdot\vec{j} = 4\pi\frac{\hbar k}{m}\delta(r).$$

P.S. for those of you who had trouble with `PlotVectorField`. You need to load the package by typing `<<Graphics‘PlotField‘` before using `PlotVectorField`. If it did not work, quitting and restarting Mathematica might do the trick.