HW #11 (221B), due Apr 22, 4pm

1. Suppose annihilation and creation operators satisfy the standard commutation relation $[a, a^\dagger] = 1$.

   (a) Show that the Bogliubov transformation
   $$b = a \cosh \eta + a^\dagger \sinh \eta \tag{1}$$
   preserves the commutation relation of creation and annihilation operators $[b, b^\dagger] = 1$.

   (b) Use this fact to obtain eigenvalues of the following Hamiltonian
   $$H = \hbar\omega a^\dagger a + \frac{1}{2} V(aa + a^\dagger a^\dagger). \tag{2}$$
   (There is an upper limit on V for which this can be done).

   (c) Show that the unitarity operator
   $$U = e^{(aa - a^\dagger a^\dagger)\eta/2} \tag{3}$$
   can relate two set of operators $b = UaU^{-1}$.

   (d) Write down the ground state of the Hamiltonian above in terms of the number states $a^\dagger a|n\rangle = n|n\rangle$.

2. We can discuss macroscopic motions of the superfluid by regarding $\psi(\vec{x}, t)$ as a classical wave with the action
   $$S = \int dt \int d\vec{x} \left[\psi^* i\hbar\dot{\psi} - \psi^* \frac{-\hbar^2 \vec{\nabla}^2}{2m} \psi + \mu \psi^* \psi - \frac{\lambda}{2} \psi^* \psi^* \psi \psi \right]. \tag{4}$$
   We are particularly interested in time-independent and z-independent solutions of the form
   $$\psi(x, y, z, t) = f(r) e^{in\theta}, \tag{5}$$
   where $f(r)$ is a real function. Answer the following questions.

   (a) Write down the velocity field $\vec{v} = \vec{\jmath}/\rho$ using the number density $\rho = \psi^* \psi$ and the momentum density $\vec{\jmath} = \frac{\hbar}{2mi}(\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi)$. Plot the velocity field for $n = 1$, -2, and 3 (*e.g.*, with `PlotVectorField`).

   (b) Show that the cirulation defined by $\kappa = \oint \vec{v} \cdot d\vec{l}$ is quantized for general n.

   (c) Write down the equation of motion in terms of $f(r)$.

   (d) Find a monotopic solution with the boundary conditions $f(0) = 0$ and $f(\infty) = \sqrt{\mu/\lambda}$ for $n = 1$. This solution is called the vortex solution.