HW #12 (221B), due Apr 29, 4pm

1. The classical energy for the Maxwell field is

$$H = \int d\vec{x} \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right). \tag{1}$$

   (a) Show that the mode expansion,

$$\begin{aligned} A^i(\vec{x}) &= \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{p}} \frac{1}{\sqrt{\omega_p}} \sum_{\pm} (\epsilon^i_\pm(\vec{p}) a_\pm(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} + \epsilon^i_\pm(\vec{p})^* a^\dagger_\pm(\vec{p}) e^{-i\vec{p}\cdot\vec{x}/\hbar}) \quad (2) \\ \dot{A}^i(\vec{x}) &= \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{p}} \sqrt{\omega_p} \sum_{\pm} (-i\epsilon^i_\pm(\vec{p}) a_\pm(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} + i\epsilon^i_\pm(\vec{p})^* a^\dagger_\pm(\vec{p}) e^{-i\vec{p}\cdot\vec{x}/\hbar}), \quad (3) \end{aligned}$$

   satisfies the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$. Here, the polarization vectors are given by

$$\begin{aligned} \vec{\epsilon}_1(\vec{p}) &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), & (4) \\ \vec{\epsilon}_2(\vec{p}) &= (-\sin\phi, \cos\phi, 0), & (5) \\ \vec{\epsilon}_\pm &= \frac{1}{\sqrt{2}}(\epsilon_1 \pm i\epsilon_2), & (6) \end{aligned}$$

   for the momentum $\vec{p} = p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

   (b) Work out the Hamiltonian using the creation and annihilation operators.

   (c) Consider a coherent state of photons in a particular momentum $\vec{p} = (0, 0, p)$ and helicity +1

$$|f\rangle = e^{-f^* f/2} e^{f a^\dagger_+(\vec{p})} |0\rangle. \tag{7}$$

   Show that the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|f\rangle = H|f\rangle$ has a solution $|f, t\rangle = |f e^{-ic|\vec{p}|t/\hbar}\rangle$. (The zero-point energy is ignored.)

   (d) Calculate the expectation value of the Maxwell field $\langle f, t|\vec{A}(\vec{x})|f, t\rangle$. You can see that this state describes a classical electromagnetic wave such as laser.