221B HW #2, due Jan 28 (Fri), 4pm

1. Solve the one-dimensional time-independent Schrödinger equation with the potential $V = \mu\delta(x)$ (μ has the dimension of energy times length).

   (a) Find the solution of the form ($k > 0$)

   $$\psi_k(x) = \begin{cases} e^{ikx} + Re^{-ikx} & (x < 0) \\ Te^{ikx} & (x > 0) \end{cases} \tag{1}$$

   R (T) is called reflection (transmission) coefficient. Verify that the unitarity relation $|R|^2 + |T|^2 = 1$.

   (b) Form the wave packet

   $$\int dq\ \psi_q(x) e^{-(q-k)^2/2\sigma^2} e^{-iE_q t/\hbar}, \tag{2}$$

   where $E_q = \hbar^2 q^2/2m$. Watch the wave packet move in Mathematica. Assume that T and R are approximately constant within the Gaussian peak $|q - k| \lesssim \sigma$.

   (c) To compare to the three-dimensional case, we can rewrite it as

   $$\psi_k(x) = e^{ikx} + \begin{cases} f(\pi) i e^{-ikx} & (x < 0) \\ f(0) i e^{ikx} & (x > 0) \end{cases} \tag{3}$$

   The "total cross section" is defined by $\sigma = |f(\pi)|^2 + |f(0)|^2$. Show the "optical theorem" $\sigma = 2\Im f(0)$ as a consequence of the unitarity relation without relying on the explicit solution.

2. Work out the probability current $\vec{j} = \frac{\hbar}{2mi}(\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi)$ for $\psi = e^{i\vec{k}\cdot\vec{x}}$ and $\psi = e^{ikr}/r$. Show that $\vec{\nabla} \cdot \vec{j}$ vanishes for the former, while it is a delta-function at the origin for the latter.

3. Consider the classical hard sphere scattering. There is a hard (impenetrable) sphere of radius a fixed at the origin. You send in a point-like particle. Calculate the differential and total cross section.