221B Lecture Notes

Notes on Spherical Bessel Functions

1 Definitions

We would like to solve the free Schrödinger equation

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}\right]R(r) = \frac{\hbar^2 k^2}{2m}R(r). \tag{1}$$

$R(r)$ is the radial wave function $\psi(\vec{x}) = R(r)Y_l^m(\theta,\phi)$. By factoring out $\hbar^2/2m$ and defining $\rho = kr$, we find the equation

$$\left[\frac{1}{\rho}\frac{d^2}{d\rho^2}\rho - \frac{l(l+1)}{\rho^2} + 1\right]R(\rho) = 0. \tag{2}$$

The solutions to this equation are spherical Bessel functions. Due to some reason, I don't see the integral representations I use below in books on mathemtical formulae, but I believe they are right.

The behavior at the origin can be studied by power expansion. Assuming $R \propto \rho^n$, and collecting terms of the lowest power in ρ, we get

$$n(n+1) - l(l+1) = 0. \tag{3}$$

There are two solutions,

$$n = l \quad \text{or} \quad -l-1. \tag{4}$$

The first solution gives a positive power, and hence a regular solution at the origin, while the second a negative power, and hence a singular solution at the origin.

It is easy to check that the following integral representations solve the above equation Eq. (2):

$$h_l^{(1)}(\rho) = -\frac{(\rho/2)^l}{l!}\int_{+1}^{i\infty} e^{i\rho t}(1-t^2)^l dt, \tag{5}$$

and

$$h_l^{(2)}(\rho) = \frac{(\rho/2)^l}{l!}\int_{-1}^{i\infty} e^{i\rho t}(1-t^2)^l dt. \tag{6}$$

By acting the derivatives in Eq. (2), one finds

$$\begin{aligned}
&\left[\frac{1}{\rho}\frac{d^2}{d\rho^2}\rho - \frac{l(l+1)}{\rho^2} + 1\right] h_l^{(1)}(\rho) \\
&= -\frac{(\rho/2)^l}{l!}\int_{\pm 1}^{i\infty}(1-t^2)^l\left[\frac{l(l+1)}{\rho^2} + \frac{2(l+1)it}{\rho} - t^2 - \frac{l(l+1)}{\rho^2} + 1\right] dt \\
&= -\frac{(\rho/2)^l}{l!}\frac{1}{i\rho}\int_{\pm 1}^{i\infty}\frac{d}{dt}\left[e^{i\rho t}(1-t^2)^{l+1}\right] dt. \qquad (7)
\end{aligned}$$

Therefore only boundary values contribute, which vanish both at $t = 1$ and $t = i\infty$ for $\rho = kr > 0$. The same holds for $h_l^{(2)}(\rho)$.

One can also easily see that $h_l^{(1)*}(\rho) = h_l^{(2)}(\rho^*)$ by taking the complex conjugate of the expression Eq. (5) and changing the variable from t to $-t$.

The integral representation Eq. (5) can be expanded in powers of $1/\rho$. For instance, for $h_l^{(1)}$, we change the variable from t to x by $t = 1 + ix$, and find

$$\begin{aligned}
h_l^{(1)}(\rho) &= -\frac{(\rho/2)^l}{l!}\int_0^\infty e^{i\rho(1+ix)}x^l(-2i)^l\left(1-\frac{x}{2i}\right)^l idx \\
&= -i\frac{(\rho/2)^l}{l!}e^{i\rho}(-2i)^l\sum_{k=0}^{l} {}_lC_k\int_0^\infty e^{-x\rho}\left(-\frac{x}{2i}\right)^k x^l dx \\
&= -i\frac{e^{i\rho}}{\rho}\sum_{k=0}^{l}\frac{(-i)^{l-k}(l+k)!}{2^k k!(l-k)!}\frac{1}{\rho^k}. \qquad (8)
\end{aligned}$$

Similarly, we find

$$h_l^{(2)}(\rho) = i\frac{e^{-i\rho}}{\rho}\sum_{k=0}^{l}\frac{i^{l-k}(l+k)!}{2^k k!(l-k)!}\frac{1}{\rho^k}. \qquad (9)$$

Therefore both $h_l^{(1,2)}$ are singular at $\rho = 0$ with power ρ^{-l-1}.

The combination $j_l(\rho) = (h_l^{(1)} + h_l^{(2)})/2$ is regular at $\rho = 0$. This can be seen easily as follows. Because $h_l^{(2)}$ is an integral from $t = -1$ to $i\infty$, while $h_l^{(1)}$ from $t = +1$ to $i\infty$, the differencd between the two corresponds to an integral from $t = -1$ to $t = i\infty$ and coming back to $t = +1$. Because the integrand does not have a pole, this contour can be deformed to a straight integral from $t = -1$ to $+1$. Therefore,

$$j_l(\rho) = \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1}^{1} e^{i\rho t}(1-t^2)^l dt. \qquad (10)$$

In this expression, $\rho \to 0$ can be taken without any problems in the integral and hence $j_l \propto \rho^l$, *i.e.*, regular. The other linear combination $n_l = (h_l^{(1)} - h_l^{(2)})/2i$ is of course singular at $\rho = 0$.[1] Note that

$$h_l^{(1)}(\rho) = j_l(\rho) + i\, n_l(\rho) \tag{11}$$

is analogous to

$$e^{i\rho} = \cos\rho + i \sin\rho. \tag{12}$$

It is useful to see some examples for low l.

$$\begin{array}{lll}
j_0 = \frac{\sin\rho}{\rho}, & j_1 = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}, & j_2 = \frac{3-\rho^2}{\rho^3}\sin\rho - \frac{3}{\rho^2}\cos\rho, \\
n_0 = -\frac{\cos\rho}{\rho}, & n_1 = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}, & n_2 = -\frac{3-\rho^2}{\rho^3}\cos\rho - \frac{3}{\rho^2}\sin\rho, \\
h_0^{(1)} = -i\frac{e^{i\rho}}{\rho}, & h_1^{(1)} = -i\left(\frac{1}{\rho^2} - \frac{i}{\rho}\right)e^{i\rho} & h_2^{(1)} = -i\left(\frac{3-\rho^2}{\rho^3} - \frac{3i}{\rho^2}\right)e^{i\rho}. \\
h_0^{(2)} = i\frac{e^{-i\rho}}{\rho}, & h_1^{(2)} = i\left(\frac{1}{\rho^2} + \frac{i}{\rho}\right)e^{-i\rho} & h_2^{(2)} = i\left(\frac{3-\rho^2}{\rho^3} + \frac{3i}{\rho^2}\right)e^{-i\rho}.
\end{array} \tag{13}$$

2 Power Series Expansion

Eq. (5) can be used to obtain the power series expansion. We first split the integration region into two parts,

$$h_l^{(1)}(\rho) = -\frac{(\rho/2)^l}{l!}\int_{+1}^{i\infty} e^{i\rho t}(1-t^2)^l dt = -\frac{(\rho/2)^l}{l!}\left[\int_0^{i\infty} - \int_0^1\right] e^{i\rho t}(1-t^2)^l dt. \tag{14}$$

The first term can be expanded in a power series by a change of variable, $t = i\tau/\rho$,

$$\begin{aligned}
\text{the first term} &= -\frac{(\rho/2)^l}{l!}\int_0^\infty e^{-\tau}\left(1+\frac{\tau^2}{\rho^2}\right)^l \frac{id\tau}{\rho} \\
&= -i\frac{1}{l!2^l\rho^{l+1}}\int_0^\infty e^{-\tau}(\tau^2+\rho^2)^l d\tau \\
&= -i\frac{1}{l!2^l\rho^{l+1}}\int_0^\infty e^{-\tau}\sum_{n=l}^{l} {}_lC_n \rho^{2n}\tau^{2l-2n} d\tau \\
&= -i\frac{1}{l!2^l\rho^{l+1}}\sum_{n=l}^{l}\frac{l!}{n!(l-n)!}\rho^{2n}\Gamma(2l-2n+1)
\end{aligned}$$

[1]Note that my notation for n_l differs from Sakurai's by a sign as seen in Eq. (7.6.52) on page 409. I'm sorry for that, but I stick with my convention, which was taken from Messiah.

$$= \quad -i\frac{1}{2^l \rho^{l+1}} \sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n}. \tag{15}$$

On the other hand, the second term can be expanded as

$$\begin{aligned}
\text{the second term} &= \frac{(\rho/2)^l}{l!} \int_0^1 e^{i\rho t}(1-t^2)^l dt \\
&= \frac{(\rho/2)^l}{l!} \int_0^1 \sum_{n=0}^{\infty} \frac{i^n}{n!} \rho^n t^n (1-t^2)^l dt \\
&= \frac{(\rho/2)^l}{l!} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^1 \rho^n t^n (1-t^2)^l dt \\
&= \frac{(\rho/2)^l}{l!} \sum_{n=0}^{\infty} \frac{i^n}{n!} \rho^n \int_0^1 x^{(n-1)/2} (1-x)^l \frac{1}{2} dx \\
&= \frac{1}{2} \frac{(\rho/2)^l}{l!} \sum_{n=0}^{\infty} \frac{i^n}{n!} \rho^n \frac{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(l+1)}{\Gamma(\frac{n}{2}+l+\frac{3}{2})}.
\end{aligned} \tag{16}$$

At this point, it is useful to separate the sum to even $n = 2k$ and odd $n = 2k+1$,

$$\begin{aligned}
&\text{the second term} \\
&= \frac{1}{2}\left(\frac{\rho}{2}\right)^l \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \rho^{2k} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+l+\frac{3}{2})} + \sum_{k=0}^{\infty} \frac{i(-1)^k}{(2k+1)!} \rho^{2k+1} \frac{\Gamma(k+1)}{\Gamma(k+l+2)} \right) \\
&= \frac{\rho^l}{2^{l+1}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \rho^{2k} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+l+\frac{3}{2})} + \sum_{k=0}^{\infty} \frac{i(-1)^k}{(2k+1)!} \rho^{2k+1} \frac{k!}{(k+l+1)!} \right)
\end{aligned} \tag{17}$$

Because $h_l^{(1)}(\rho) = j_l(\rho) + i n_l(\rho)$, we find

$$j_l(\rho) = \frac{\rho^l}{2^{l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+l+\frac{3}{2})} \rho^{2k} \tag{18}$$

$$n_l(\rho) = -\frac{1}{2^l \rho^{l+1}} \sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n} + \frac{\rho^l}{2^{l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!(k+l+1)!} \rho^{2k+1}. \tag{19}$$

The expression for j_l can be simplified using the identity $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$,

$$j_l(\rho) = \frac{\rho^l}{2^{l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k-1)!!\sqrt{\pi}/2^k}{(2k+2l+1)!!\sqrt{\pi}/2^{k+l+1}} \rho^{2k}$$

$$
\begin{aligned}
&= \rho^l \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{(2k+2l+1)(2k+2l-1)\cdots(2k+1)} \rho^{2k} \\
&= \rho^l \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k+2l)(2k+2l-2)\cdots(2k+2)(2k)!}{(2k+2l+1)!} \rho^{2k} \\
&= \rho^l \sum_{k=0}^{\infty} \frac{(-1)^k 2^l (k+l)!}{k!(2k+2l+1)!} \rho^{2k} \\
&= (2\rho)^l \sum_{k=0}^{\infty} \frac{(-1)^k (k+l)!}{k!(2k+2l+1)!} \rho^{2k}. \qquad (20)
\end{aligned}
$$

To write out the first three terms,

$$
= \frac{\rho^l}{(2l+1)!!} \left[1 - \frac{\rho^2}{2(2l+3)} + \frac{\rho^4}{8(2l+5)(2l+3)} - \cdots \right]. \qquad (21)
$$

It suggests that the leading term is a good approximation when $\rho \ll 2l^{1/2}$.

Similarly, the expression for $n_l(\rho)$ can also be simplified,

$$
\begin{aligned}
n_l(\rho) &= -\frac{1}{2^l \rho^{l+1}} \sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n} + \frac{\rho^l}{2^{l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!(k+l+1)!} \rho^{2k+1} \\
&= -\frac{1}{2^l \rho^{l+1}} \left(\sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!(k+l+1)!} \rho^{2k+2l+2} \right) \\
&= -\frac{1}{2^l \rho^{l+1}} \left(\sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n} + \frac{1}{2} \sum_{n=l+1}^{\infty} \frac{(-1)^{n+l}(n-l-1)!}{(2n-2l-1)!n!} \rho^{2n} \right) \\
&= -\frac{1}{2^l \rho^{l+1}} \left(\sum_{n=l}^{l} \frac{(2l-2n)!}{n!(l-n)!} \rho^{2n} + \sum_{n=l+1}^{\infty} \frac{(-1)^{n+l}(n-l)!}{(2n-2l)!n!} \rho^{2n} \right) \\
&= -\frac{1}{2^l \rho^{l+1}} \sum_{n=l} \frac{(-1)^n}{n!} \frac{\Gamma(2l-2n+1)}{\Gamma(l-n+1)} \rho^{2n}. \qquad (22)
\end{aligned}
$$

To write out the first three terms,

$$
= -\frac{(2l-1)!!}{\rho^{l+1}} \left[1 + \frac{\rho^2}{2(2l-1)} + \frac{\rho^4}{8(2l-1)(2l-3)} + \cdots \right]. \qquad (23)
$$

It suggests that the leading term is a good approximation when $\rho \ll 2l^{1/2}$.

3 Asymptotic Behavior

Eqs. (8,9) give the asymptotic behaviors of $h_l^{(1)}$ for $\rho \to \infty$:

$$h_l^{(1)} \sim -i\frac{e^{i\rho}}{\rho}(-i)^l = -i\frac{e^{i(\rho - l\pi/2)}}{\rho}. \tag{24}$$

By taking linear combinations, we also find

$$j_l \sim \frac{\sin(\rho - l\pi/2)}{\rho}, \tag{25}$$

$$n_l \sim -\frac{\cos(\rho - l\pi/2)}{\rho}. \tag{26}$$

These expressions are good approximations when $\rho \gg l^2$. As seen in the next section, there are better approximations when $\rho \geq l \geq 1$.

4 Large l Behavior

Starting from the integral from Eq. (10), we use the steepest descent method to find the large l behavior. Changing the variable $t = l\tau$,

$$\begin{aligned}
j_l(\rho) &= \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1}^{1} e^{i\rho t}(1-t^2)^l dt \\
&= \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1/l}^{1/l} e^{l(\log(1-l^2\tau^2)+i\rho\tau)} l d\tau \\
&= \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1/l}^{1/l} \exp\left[-l + \sqrt{l^2-\rho^2} + l\log\frac{2l(l-\sqrt{l^2-\rho^2})}{\rho^2}\right. \\
&\qquad \left. -\frac{l\rho^2\sqrt{l^2-\rho^2}}{2(l-\sqrt{l^2-\rho^2})}\left(\tau - i\frac{l-\sqrt{l^2-\rho^2}}{l\rho}\right)^2 + O(\Delta\tau)^3\right] l d\tau \\
&\simeq \frac{1}{2}\frac{(\rho/2)^l}{\sqrt{2\pi l}\, l^l e^{-l}} e^{-l} e^{\sqrt{l^2-\rho^2}} \left(\frac{2l(l-\sqrt{l^2-\rho^2})}{\rho^2}\right)^l \left(\frac{2\pi(l-\sqrt{l^2-\rho^2})}{l\rho^2\sqrt{l^2-\rho^2}}\right)^{1/2} l \\
&= \frac{1}{2\rho} e^{\sqrt{l^2-\rho^2}} \left(\frac{l-\sqrt{l^2-\rho^2}}{\rho}\right)^l \left(\frac{l-\sqrt{l^2-\rho^2}}{\sqrt{l^2-\rho^2}}\right)^{1/2}.
\end{aligned} \tag{27}$$

This expression works very well as long as $l \gg 1$ and $\rho \leq l$.

Starting from the integral from Eq. (5), we use the steepest descent method to find the large l behavior. Change the variable $t = il\tau$,

$$
\begin{aligned}
h_l^{(1)}(\rho) &= -\frac{(\rho/2)^l}{l!}\int_{+1}^{i\infty} e^{i\rho t}(1-t^2)^l dt \\
&= -il\frac{(\rho/2)^l}{l!}\int_{-i/l}^{\infty} \exp\left[-l-\sqrt{l^2-\rho^2}+l\log\frac{2l(l+\sqrt{l^2-\rho^2})}{\rho^2}\right. \\
&\qquad \left. -\frac{l\rho^2\sqrt{l^2-\rho^2}}{2(l+\sqrt{l^2-\rho^2})}\left(\tau-\frac{l+\sqrt{l^2-\rho^2}}{l\rho}\right)^2+O(\Delta\tau)^3\right] \\
&\simeq -ie^{-\sqrt{l^2-\rho^2}}\left(\frac{l+\sqrt{l^2-\rho^2}}{\rho}\right)^l\left(\frac{l+\sqrt{l^2-\rho^2}}{\rho^2\sqrt{l^2-\rho^2}}\right)^{1/2}. \qquad (28)
\end{aligned}
$$

Note that there are actually two saddle points,

$$
\tau = \frac{l \pm \sqrt{l^2-\rho^2}}{l\rho}. \qquad (29)
$$

In the above calculation, we picked the saddle point with the negative sign with the steepest descent, while the other saddle point is what we picked for $j_l(\rho)$. Therefore,

$$
n_l(\rho) \simeq -\frac{1}{\rho}e^{-\sqrt{l^2-\rho^2}}\left(\frac{l+\sqrt{l^2-\rho^2}}{\rho}\right)^l\left(\frac{l+\sqrt{l^2-\rho^2}}{\sqrt{l^2-\rho^2}}\right)^{1/2}. \qquad (30)
$$

This expression again works very well as long as $l \gg 1$ and $\rho \leq l$ (but not too close).

On the other hand, for $\rho \geq l \gg 1$, the saddle points above become complex. The contribution to the $h_l^{(1)}(\rho)$ is given by the saddle point $\tau = \frac{l-i\sqrt{\rho^2-l^2}}{l\rho}$, and hence

$$
h_l^{(1)}(\rho) \simeq \frac{1}{\rho}e^{i\sqrt{\rho^2-l^2}}\left(\frac{l-i\sqrt{\rho^2-l^2}}{\rho}\right)^l\left(\frac{l-i\sqrt{\rho^2-l^2}}{i\sqrt{\rho^2-l^2}}\right)^{1/2}. \qquad (31)
$$

This works very well as long as $l \gg 1$ and $\rho \geq l$ (but not too close). j_l (n_l) is given by the real (imaginary) part of $h_l^{(1)}(\rho)$. In practice, this form works remarkably well even for $l = 1$.

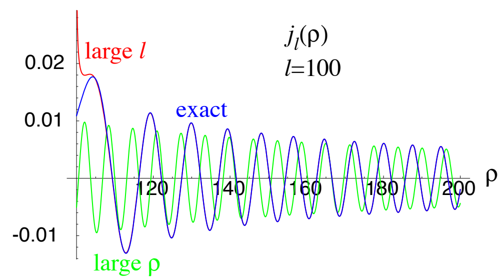

Figure 1: Comparison of the large ρ behavior and the large l behavior of $j_l(\rho)$ to the exact result. The large l behavior is a very good approximation for $\rho \geq 105 > 100 = l$, while the large ρ behavior is still a poor approximation unless $\rho > O(l^2)$.

It is interesting to note that this asymptotic behavior of $h_l^{(1)}(\rho)$ is what you expect from the semi-classical approximation for the free-particle wave function. The classical action for a free particle is

$$S(r) = \hbar \int_{l/k}^{r} \sqrt{k^2 - \frac{l^2}{r'^2}}\, dr' = \hbar\sqrt{(kr)^2 - l^2} \; - 2l \arctan\sqrt{\frac{kr-l}{kr+l}} \tag{32}$$

and hence

$$e^{iS(r)/\hbar} = e^{i\sqrt{(kr)^2-l^2}} \left(\frac{l - i\sqrt{(kr)^2 - l^2}}{kr} \right)^l, \tag{33}$$

which agrees with the large l behavior above except for the last factor which comes from the lowest-order quantum correction.

When $\rho \simeq l \gg 1$, two saddle points collide and I don't know what to do.

5 Recursion Formulae

Starting from Eq. (5), we take the derivative

$$\frac{d}{d\rho} h_l^{(1)} = \frac{l}{\rho} h_l^{(1)} - \frac{(\rho/2)^l}{l!} \int_{+1}^{i\infty} e^{i\rho t} it(1-t^2)^l dt. \tag{34}$$

The second term can be integrated by parts, and gives

$$= \frac{l}{\rho} h_l^{(1)} + \frac{\rho}{2} \frac{(\rho/2)^l}{l!} \int_{+1}^{i\infty} e^{i\rho t} (1-t^2)^{l+1} dt = \frac{l}{\rho} h_l^{(1)} - h_{l+1}^{(1)}. \tag{35}$$

In fact, other functions j_l, n_l, and $h_l^{(2)}$ all satisfy the same relation which can be easily checked. Referring to all of them generically as $z_l(\rho)$, we find the recursion formula

$$z_l' = \frac{l}{\rho} z_l - z_{l+1}. \tag{36}$$

Because $z_l(\rho)$ satisfies the differential equation Eq. 2, we can combine it with the above recursion relation and find

$$\begin{aligned} 0 &= \left(\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{l(l+1)}{\rho^2} + 1 \right) z_l \\ &= z_l - \frac{2l+3}{\rho} z_{l+1} + z_{l+2}. \end{aligned} \tag{37}$$

Relabeling l to $l-1$, we obtain

$$z_{l-1} + z_{l+1} = \frac{2l+1}{\rho} z_l. \tag{38}$$

Finally, combining the two recursion relations, we also obtain

$$z_l' = z_{l-1} - \frac{l+1}{\rho} z_l. \tag{39}$$

6 Plane Wave Expansion

The non-trivial looking formula we used in the class

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta) \tag{40}$$

can be obtained quite easily from the integral representation Eq. (10). The point is that one can keep integrating it in parts. By integrating $e^{i\rho t}$ factor and differentiating $(1-t^2)^l$ factor, the boundary terms at $t = \pm 1$ always vanish up to l-th time because of the $(1-t^2)^l$ factor. Therefore,

$$j_l = \frac{1}{2} \frac{(\rho/2)^l}{l!} \int_{-1}^{1} \frac{1}{(i\rho)^l} e^{i\rho t} \left(-\frac{d}{dt} \right)^l (1-t^2)^l dt. \tag{41}$$

Note that the definition of the Legendre polynomials is

$$P_l(t) = \frac{1}{2^l} \frac{1}{l!} \frac{d^l}{dt^l} (t^2-1)^l. \tag{42}$$

Using this definition, the spherical Bessel function can be written as

$$j_l = \frac{1}{2}\frac{1}{i^l}\int_{-1}^{1} e^{i\rho t} P_l(t) dt. \tag{43}$$

Then we use the fact that the Legendre polynomials form a complete set of orthogonal polynomials in the interval $t \in [-1, 1]$. Noting the normalization

$$\int_{-1}^{1} P_n(t) P_m(t) dt = \frac{2}{2n+1}\delta_{n,m}, \tag{44}$$

the orthonormal basis is $P_n(t)\sqrt{(2n+1)/2}$, and hence

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(t) P_n(t') = \delta(t - t'). \tag{45}$$

By multipyling Eq. (43) by $P_l(t')(2l+1)/2$ and summing over n,

$$\sum_{l=0}^{\infty} 2i^l \frac{2l+1}{2} P_l(t') j_l(\rho) = \int_{-1}^{1} e^{i\rho t} \sum_{n=0}^{\infty} \frac{2l+1}{2} P_l(t') P_l(t) dt = e^{i\rho t'}. \tag{46}$$

By setting $\rho = kr$ and $t' = \cos\theta$, we prove Eq. (40).

If the wave vector is pointing at other directions than the positive z-axis, the formula Eq. (40) needs to be generalized. Noting $Y_l^0(\theta, \phi) = \sqrt{(2l+1)/4\pi}\, P_l(\cos\theta)$, we find

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^{m*}(\theta_{\vec{k}}, \phi_{\vec{k}}) Y_l^m(\theta_{\vec{x}}, \phi_{\vec{x}}) \tag{47}$$

7 Delta-Function Normalization

An important consequence of the identity Eq. (47) is the innerproduct of two spherical Bessel functions. We start with

$$\int d\vec{x} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k} - \vec{k}'). \tag{48}$$

Using Eq. (47) in the l.h.s of this equation, we find

$$\begin{aligned}
&\int d\vec{x} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}} \\
&= \sum_{l,m}\sum_{l',m'} (4\pi)^2 \int d\Omega_{\vec{x}} dr r^2 Y_l^{m*}(\Omega_{\vec{k}}) Y_l^m(\Omega_{\vec{x}}) Y_{l'}^{m'*}(\Omega_{\vec{x}}) Y_{l'}^{m'}(\Omega_{\vec{k}'}) j_l(kr) j_{l'}(k'r) \\
&= \sum_{l,m} (4\pi)^2 \int dr r^2 j_l(kr) j_l(k'r) Y_l^{m*}(\Omega_{\vec{k}}) Y_l^m(\Omega_{\vec{k}'}). \qquad (49)
\end{aligned}$$

On the other hand, the r.h.s. of Eq. (48) is

$$(2\pi)^3\delta(\vec{k}-\vec{k}') = (2\pi)^3\frac{1}{k^2}\delta(k-k')\delta(\Omega_{\vec{k}}-\Omega_{\vec{k}'}) = (2\pi)^3\frac{1}{k^2\sin\theta}\delta(k-k')\delta(\theta-\theta')\delta(\phi-\phi'). \quad (50)$$

Comparing Eq. (49) and (50) and noting

$$\sum_{l,m} Y_l^{m*}(\Omega_{\vec{k}})Y_l^m(\Omega_{\vec{k}'}) = \delta(\Omega_{\vec{k}}-\Omega_{\vec{k}'}), \quad (51)$$

we find

$$\int_0^\infty drr^2 j_l(kr)j_l(k'r) = \frac{\pi}{2k^2}\delta(k-k'). \quad (52)$$

8 Mathematica

In Mathematica, spherical Bessel functions are not defined but the usual Bessel functions are. The $j_l(z)$ is obtained by

$$\sqrt{\frac{\pi}{2\texttt{z}}}\texttt{BesselJ}[\texttt{l}+\frac{1}{2},\texttt{z}]$$

and $n_l(z)$ by

$$\sqrt{\frac{\pi}{2\texttt{z}}}\texttt{BesselY}[\texttt{l}+\frac{1}{2},\texttt{z}].$$

You may actually want to use

$$\texttt{PowerExpand}[\sqrt{\frac{\pi}{2\texttt{z}}}\texttt{BesselJ}[\texttt{l}+\frac{1}{2},\texttt{z}]]$$

etc to get rid of half-odd powers.