221B Lecture Notes
Scattering Theory II

1 Born Approximation

Lippmann–Schwinger equation

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V|\psi\rangle, \tag{1}$$

is an exact equation for the scattering problem, but it still is an equation to be solved because the state vector $|\psi\rangle$ appears on both sides of the equation. In the coordinate space, as we derived in Scattering Theory I, it becomes

$$\psi(\vec{x}) \simeq \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} e^{-i\vec{k'}\cdot\vec{x'}} V(\vec{x'})\psi(\vec{x'}), \tag{2}$$

far away from the scatterer where $r = |\vec{x}|$ and $\vec{k'} = \vec{k}\frac{\vec{x}}{r}$ is the wave-vector of the scattered wave. Note that $|\vec{k'}| = \vec{k}$. It is an integral equation for the unknown function $\psi(\vec{x})$.

One way to solve the Lippmann–Schwinger equation Eq. (1) is by perturbation theory, *i.e.*, a power series expansion in the potential V. Note that, in the absence of the potential, $|\psi\rangle = |\phi\rangle$, or in other words, $|\psi\rangle = |\phi\rangle + O(V)$. Therefore the lowest (1st) order approximation in V is write

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V|\phi\rangle + O(V^2), \tag{3}$$

and neglect $O(V^2)$ correction. This is called *Born approximation*,[1] or more correctly, 1st Born approximation. Obviously, this approximation is good only when the scattering is weak.

In the coordinate space, we again replace ψ by ϕ in the r.h.s. of Eq. (2), and find

$$\begin{aligned}\psi(\vec{x}) &\simeq \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} e^{-i\vec{k'}\cdot\vec{x'}} V(\vec{x'}) \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x'}} \\ &= \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} V(\vec{x'}) e^{i\vec{q}\cdot\vec{x'}}\right],\end{aligned} \tag{4}$$

[1] Did you know that Max Born was the grandfather of Olivia Newton-John? See, *e.g.*, `http://mooni.fccj.org/~ethall/trivia/trivia.htm`.

where $\vec{q} = \vec{k} - \vec{k'}$ is the *momentum transfer* in the scattering process.

The expression Eq. (4) is very interesting. It shows that the scattering amplitude is the Fourier transform of the potential,

$$f^{(1)}(\vec{k'}, \vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\vec{x} V(\vec{x}) e^{i\vec{q}\cdot\vec{x}}, \tag{5}$$

up to a numerical factor of $-(1/4\pi)(2m/\hbar^2)$. The superscript shows that this is a result valid at the first order in V. This expression demonstrates the uncertainty principle: to probe small-scale structure of an object, you need to have a scattering experiment with a high momentum transfer, because the Fourier transform averages out small-scale structure otherwise.

If the potential is central, *i.e.*, $V(\vec{x})$ is a function of $r = |\vec{x}|$ only. Then the expression Eq. (5) can be further simplified:

$$\begin{aligned} f^{(1)}(\vec{k'}, \vec{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\cos\theta d\phi r^2 dr V(r) e^{iqr\cos\theta} \\ &= -\frac{m}{\hbar^2}\int_0^\infty dr r^2 V(r) \frac{e^{iqr} - e^{-iqr}}{iqr} \\ &= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr r V(r) \sin qr. \end{aligned} \tag{6}$$

Therefore the scattering amplitude depends only on $q = |\vec{q}| = |\vec{k} - \vec{k'}| = 2k\sin(\theta/2)$. In other words, it is a function of the polar angle θ only $f(\vec{k'}, \vec{k}) = f(\theta)$. This is a statement independent of Born approximation.

2 Rutherford Scattering

Rutherford is of course famous for his discovery that the atoms consist of electrons and a concentrated positive electric charge that we now call nuclei. He made this discovery by bombarding the α-particle on a gold foil, and looking for events where the α-particle is scattered by a large angle. Apparently he suggested a poor student Marsden to do this search thinking that he would never find one.[2] Great discoveries can't be planned.

[2] See, *e.g.*, `http://galileo.phys.virginia.edu/classes/252/Rutherford_Scattering/Rutherford_Scattering.html`

2.1 Point Coulomb Source

One of the most important application of the Born approximation is to the Coulomb potential, because this is the relevant one for the Rutherford scattering experiment. By taking

$$V(r) = \frac{ZZ'e^2}{r}, \tag{7}$$

where I took the unit where $4\pi\epsilon_0 = 1$, we would like to calculate the differential cross section. Z is the charge of the scatterer (say, gold nucleus) and Z' that of the incident particle (say, α particle). However, the expression Eq. (6) does not converge. Therefore, we start with a short-range potential called *Yukawa potential*

$$V(r) = V_0 \frac{e^{-\mu r}}{r}, \tag{8}$$

and take the limit $\mu \to 0$ to recover the Coulomb potential at the end of the calculations.[3] The Yukawa potential is a typical example of a short-ranged potential because it goes rapidly to zero once $r \gtrsim 1/\mu$. It is of great interest on its own apart from the limit $\mu \to 0$. The potential that binds protons and nucleons (nuclear force, or strong interaction) can be approximated by this type of potential, because the range of the nuclear force is only about 10^{-12} cm at most.

The formula Eq. (6) tells us that the scattering amplitude for the Yukawa potential Eq. (8) is

$$f(\theta) = -\frac{2mV_0}{\hbar^2} \frac{1}{q^2 + \mu^2}. \tag{9}$$

Different cross section is therefore given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta) + \mu^2]^2}. \tag{10}$$

The total cross section is obtained by integrating over $d\Omega = d\cos\theta d\phi$,

$$\sigma = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{4\pi}{4k^2\mu^2 + \mu^4}. \tag{11}$$

[3]My normalization of V_0 is different from J.J. Sakurai by a factor of μ, so that $\mu \to 0$ limit is taken more easily.

We can now take the limit $\mu \to 0$ and $V_0 = ZZ'e^2$ to obtain results for the Coulomb potential,

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZZ'e^2}{\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)]^2} = \frac{(2m)^2(ZZ'e^2)^2}{16(\hbar k)^4 \sin^4(\theta/2)}. \tag{12}$$

On the other hand, the total cross section Eq. (11) diverges! The divergence is in the $\cos\theta$ integral when $\theta \to 0$. In other words, the divergence occurs for the small momentum transfer $q \to 0$, which corresponds to large distances.

This result for the Coulomb scattering is exactly the same as in the classical theory by identifying $\hbar k$ as the momentum of the incident particle. It is surprising that the Born approximation actually gives an exact result for the Coulomb potential, and it agrees with the classical calculation as well. This should be considered as a coincidence because there is no reason why any of them should come out to be the same.

The reason why the total cross section diverges is because the Coulomb potential is actually a *long-range* force. No matter how far the incident particles are from the charge, there is always an effect on the motion of the particles and they get scattered.

2.2 Form Factor

At much higher momentum transfers, the α-particle even starts to resolve the charge distribution of the nucleus $\rho_N(\vec{x})$. The Coulomb potential is modified to

$$V(\vec{x}) = \int d\vec{x'} \frac{Z'e^2}{|\vec{x} - \vec{x'}|} \rho_N(\vec{x'}). \tag{13}$$

Note that the potential is mathematically a convolution of the Coulomb potential and the probability density. Since the first Born amplitude is nothing but the Fourier transform of the potential, the convolution becomes a product of Fourier transforms, one for the Coulomb potential and the other for the probability density. Indeed, after performing the integral in Eq. (6), we find

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{Z'e^2}{q^2}\int d\vec{x}\rho_N(\vec{x})e^{i\vec{q}\cdot\vec{x}} = f(\theta)_{\text{pointlike}} F(q), \tag{14}$$

where

$$f(\theta)_{\text{pointlike}} = -\frac{2m}{\hbar^2}\frac{ZZ'e^2}{q^2}, \tag{15}$$

$$F(q) = \frac{1}{Z}\int d\vec{x}\rho_N(\vec{x})e^{i\vec{q}\cdot\vec{x}}. \tag{16}$$

Clearly $f(\theta)_{\text{pointlike}}$ is the scattering amplitude for the point-like Coulomb source, namely $\mu \to 0$ limit in Eq. (9). The second factor $F(q)$ is called the form factor which depends on the charge distribution of the nucleus. In the limit $\vec{q} \to 0$, $e^{i\vec{q}\cdot\vec{x}} = 1$ and hence $F(q) = 1$; namely the momentum transfer is too low to resolve the detailed structure of the nucleus. On the other hand, for large q, $F(q)$ becomes much less than unity due to the rapidly oscillating integrand and the cross section gets suppressed.

The differential cross section reduces to the form

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{\text{pointlike}} |F(q)|^2. \tag{17}$$

In fact, Rutherford experiment already showed the deviation from the point-like Coulomb source at high momentum transfer (large angle scattering), which led him to estimate the size of the nucleus.

Fig. 1 shows the form factor $|F(q)|^2$ in an electron-nucleus scattering experiment. The oscillatory behavior can be understood qualitatively in the following way. Imagine a sphere of radius a with a uniform charge density ρ_0 such that $Z = \frac{4\pi}{3}a^3\rho_0$. The form factor, the Fourier transform, is given by

$$\begin{aligned} F(q) &= \frac{1}{Z}\int d\vec{x}\rho_N(\vec{x})e^{i\vec{q}\cdot\vec{x}} && (18)\\ &= \frac{1}{Z}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^a r^2 dr \rho_0 e^{iqr\cos\theta} && (19)\\ &= \frac{1}{Z}2\pi\int_0^a r^2 dr\rho_0 \frac{e^{iqr} - e^{-iqr}}{iqr} && (20)\\ &= 3\frac{\sin aq - aq\cos aq}{(aq)^3}. && (21) \end{aligned}$$

One can verify that $F(0) = 1$. On the other hand, this function goes down as $1/q^2$ at large q, while it oscillates in the numerator. It oscillates because the Fourier transform depends sensitively on how many waves fit inside the nucleus. The true charge density distribution is not sharply cutoff as a uniform sphere, but somewhat smoothed out at the edge, but still similar. Fourier transform of the measured form factor determined the true charge density distribution inside the nucleus, as seen in Fig. 2

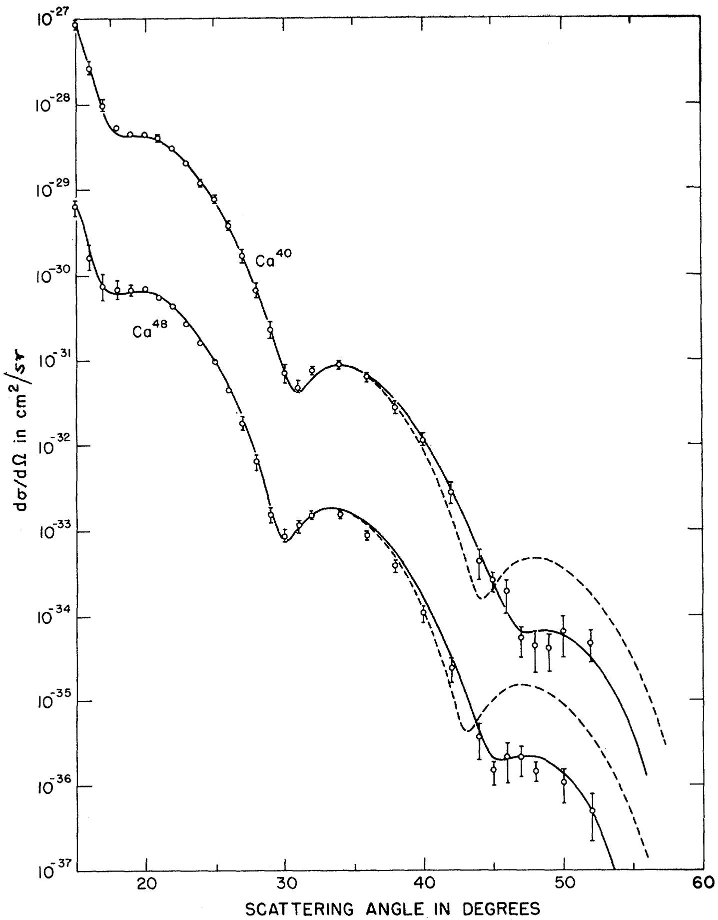

FIG. 1. Experimental and theoretical differential cross sections at 757.5 MeV. The nominal energy was 750 MeV, and a 1% adjustment was made to improve the fit at low q. The dashed curves are the best fits to earlier 250-MeV data. The charge distributions which yield them are parabolic Fermi (three-parameter) shapes [see Eq. (3) of Ref. 1] with the following parameter values: Ca^{40}, $c=3.6685$ F, $z=0.5839$ F, $w=-0.1017$; Ca^{48}, $c=3.7369$ F, $z=0.5245$ F, $w=-0.0300$. The solid curves, obtained by the method described in this Letter, come from charge distributions with an added $\Delta\rho(r)$, and parameter values $p=0.5\ \mathrm{F}^{-1}$, $q_0=3.0\ \mathrm{F}^{-1}$, and $A(Ca^{40})=0.5\times10^{-3}$, $A(Ca^{48})=0.8\times10^{-3}$. The cross section for Ca^{40} has been multiplied by 10 and that for Ca^{48} by 10^{-1}.

Figure 1: Elastic electron scattering off calcium. Taken from J. B. Bellicard et al, *Phys. Rev. Lett.*, **19**, 527 (1967)

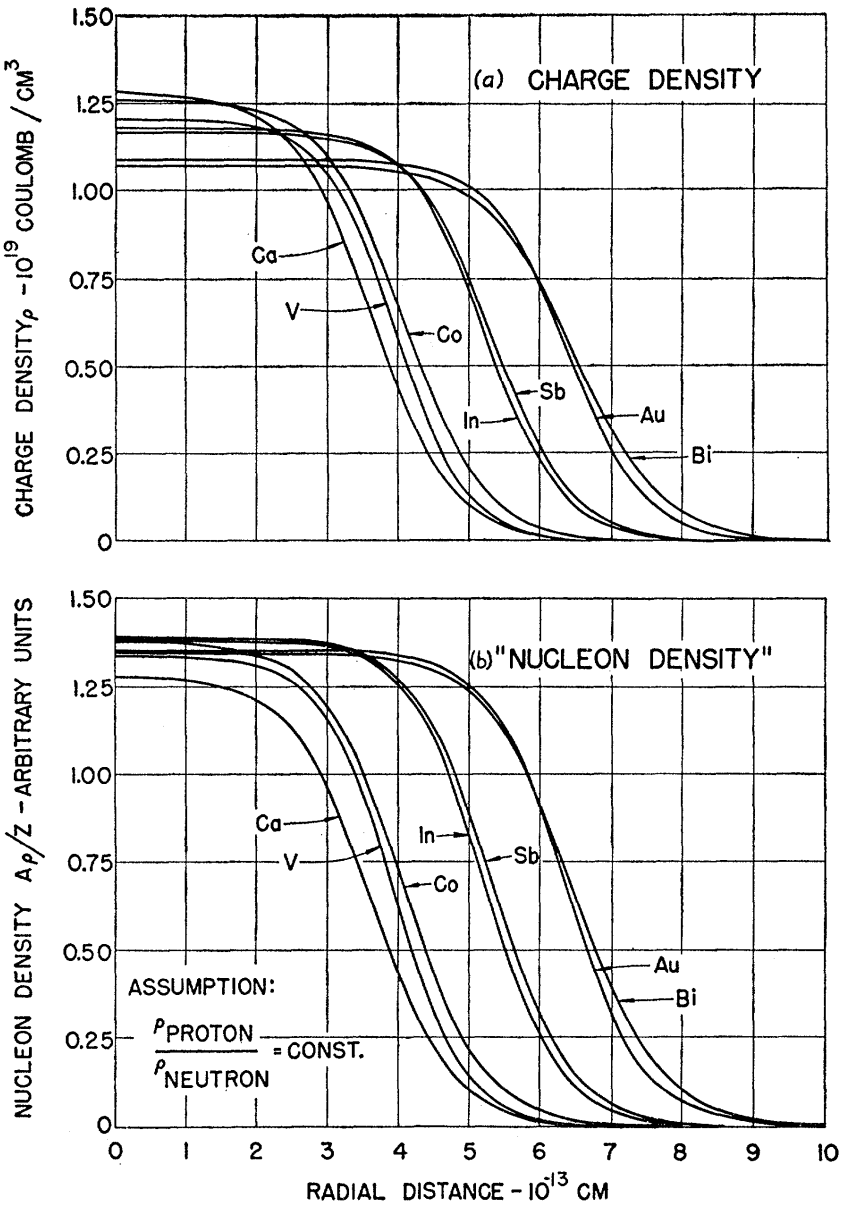

FIG. 14. (a) Charge distributions $\rho(r)$ for Ca, V, Co, In, Sb, Au, and Bi. They are Fermi smoothed uniform shapes, with the parameters given in Table III, and yield the cross sections shown in Figs. 3 and 8–12. (b) A plot of $(A/2Z)\rho(r)$ for the above nuclei. On the assumption that the distribution of matter in the nucleus is the same as the distribution of charge, this represents the "nucleon density."

Figure 2: Taken from B. Hahn, D. G. Ravenhall, and R. Hofstadter, *Phys. Rev.* **101**, 1131-1142 (1956).

Thanks to these experiments, we have learned that the nuclei are more or less a spherical ball of fixed density, and hence the size of the nucleus scales as $A^{1/3}$ as a function of the mass number $A = Z + N$. Nuclei are very small, with a radius of approximately 1.12 fm $\times A^{1/3}$. Note that fm $= 10^{-15}$m, and hence 10^4–10^5 times smaller than the Bohr radius.

Later, much more precise and higher energy electron-proton scattering experiments were performed, which showed that the form factor has an approximate dipole form (Fig. 3)

$$F(q) \simeq \frac{1}{(1 + q^2 a_N^2)^2}, \tag{22}$$

where $a_N \simeq 0.26$ fm. From the inverse Fourier transform, one can see that the charge density of the proton has approximately an exponential profile $\propto e^{-r/a_N}$. This is probably one of the earliest evidences for the composite nature of the proton.

The form factor can be used to also study the effect of the electrons in the atom on the Rutherford scattering. Let us go back to smaller momentum transfer so that the nucleus is seen as a point-like Coulomb source. We expect that the electrons screen the charge of the nucleus at large radii and hence makes the total cross section finite. What would be the cross section in that case? The Coulomb potential then is modified at long distances (distance beyond Bohr radius) where

$$V(\vec{x}) = \frac{ZZ'e^2}{|\vec{x}|} - \int d\vec{x'} \frac{Z'e^2}{|\vec{x} - \vec{x'}|} \rho(\vec{x'}), \tag{23}$$

where $\rho(\vec{x'})$ is the probability density of the electron cloud with the normalization $\int d\vec{x'} \rho(\vec{x'}) = Z$. $\rho(\vec{x'})$ is concentrated within the size of the atom $|\vec{x'}| \lesssim a$. Very far away from the atom, the second term cancels the first term and there is no potential.

Note that the second term is basically a convolution of the Coulomb potential and the probability density. Since the first Born amplitude is nothing but the Fourier transform of the potential, the convolution becomes a product of Fourier transforms, one for the Coulomb potential and the other for the probability density. Indeed, after performing the integral in Eq. (6), we find

$$f(\theta) = -\frac{2m}{\hbar^2} \frac{ZZ'e^2}{q^2} \left[1 - \frac{1}{Z} \int d\vec{x} \rho(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \right]. \tag{24}$$

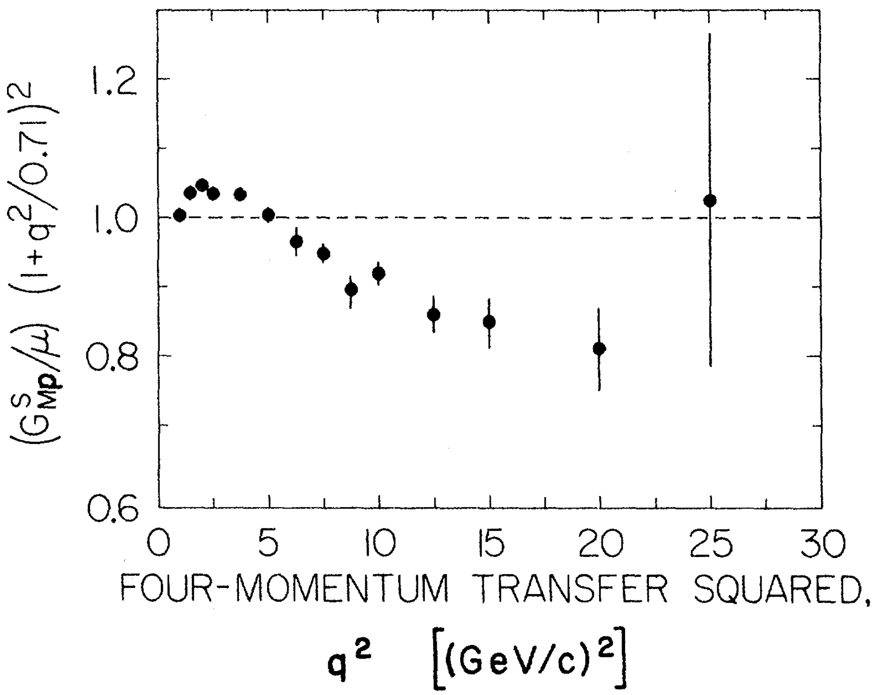

FIG. 18. The quantity $(G^S_{Mp}/\mu)(1+q^2/0.71)^2 = G^S_{Ep}/G^D_{Ep}$ from the present experiment as a function of q^2. The estimated normalization error of $\pm 4\%$ is not included. The measured form factor seems to fall faster than $1/q^4$ at high q^2.

Figure 3: Elastic electron-proton scattering cross section compared to the dipole form factor. Taken from P.N. Kirk *et al*, *Phys. Rev.* **D 8**, 63 (1973).

In the limit $\vec{q} \to 0$, where the cross section diverges, two terms in the square bracket cancel because the second term approaches unity.

To gain more insight, let us take a simple case of the hydrogen atom $Z = 1$. The electron wave function in the ground state is

$$\psi(\vec{x}) = \frac{1}{\sqrt{4\pi}} 2a^{-3/2} e^{-r/a}. \tag{25}$$

$a = \hbar^2/me^2$ is the Bohr radius. The probability density of the electron cloud is then

$$\rho(\vec{x}) = |\psi(\vec{x})|^2 = \frac{1}{\pi a^3} e^{-2r/a}. \tag{26}$$

All we need to know now is the Fourier transform of this probability density. It is straightforward to obtain

$$\int d\vec{x} \rho(\vec{x}) e^{i\vec{q}\cdot\vec{x}} = \frac{16}{(4 + q^2a^2)^2}. \tag{27}$$

For $\vec{q} \to 0$, the l.h.s. is simply the normalization of the wave function, *i.e.*, unity. The r.h.s. indeed gives the same limit. On the other hand, it vanishes when $q \gg a^{-1}$. In other words, for momentum transfer larger than the inverse size of the atom $\hbar/a$, the electron cloud does not change the cross section from the case of a point Coulomb source.

Eq. (24) is now given by

$$f(\theta) = -\frac{2m}{\hbar^2} Z' e^2 a^2 \frac{8 + 4(qa)^2}{(4 + (qa)^2)^2}. \tag{28}$$

When $q \to 0$, the amplitude is regular and the total cross section converges. Recalling $q^2 = 2k^2(1 - \cos\theta)$, we find

$$\sigma = \int d\Omega |f(\theta)|^2 = 2\pi \left(\frac{2m}{\hbar^2} Z' e^2 a^2\right)^2 \frac{-(k^2a^2) + 2(1 + k^2a^2)\log(1 + k^2a^2)}{k^2a^2 + k^4a^4} \tag{29}$$

For small $k \ll a^{-1}$, the last factor becomes unity, and the total cross section is

$$\sigma(k = 0) = 2\pi \left(\frac{2m}{\hbar^2} Z' e^2 a^2\right)^2 = 8\pi Z'^2 \left(\frac{m}{m_e}\right)^2 a^2. \tag{30}$$

However, this result cannot be true. The geometric cross section of the target (the atom) is only of the order of πa^2. Because $m \gg m_e$, this total cross section is far larger than the geometric cross section. It signals the breakdown of perturbation theory: the Born approximation is invalid. Using the discussion of the validity in the next section, one can also see explicitly why that is the case. On the other hand, for a high momentum $k \gg a^{-1}$,

$$\sigma \simeq 8\pi Z'^2 \left(\frac{m}{m_e}\right)^2 a^2 \frac{2\log(1+k^2a^2)-1}{k^2a^2}. \tag{31}$$

As long as $k \gg a^{-1}(m/m_e)$, Born approximation is valid and the total cross section can be trusted.

2.3 Coulomb Wave Function

Back to the point-like Coulomb source, we obtained the Rutherford formula with the 1st Born approximation, which agrees with purely classical result. We have also seen that the long-range nature of the Coulomb potential actually results in an infinite total cross section. The long-range nature, however, causes another problem. Because the α-particle (or any charged particle for that matter) feels the Coulomb potential no matter how far it is, the incident wave can never be described accurately by a plane wave. In fact, there is a logarithmic correction to it. We do not go into the discussion in detail in this lecture note, but you can look at the very last section of Sakurai on this issue. Instead of plane waves, you are supposed to use the Coulomb wave functions, which are the exact solutions to the Schrödinger equation in the presence of the Coulomb energy with positive energies (non-bound and hence continuum state) with a definite angular momentum.

Fortunately, the result obtained from the exact solution turns out to agree both with the Born approximation and the classical result. Rutherford was lucky in many ways.

3 Born Expansion

Of course, the first Born approximation is only the leading order in V. We can work out higher orders from Eq. (3), by iteratively insert the r.h.s. of the equation at a given order in V back into the $|\psi\rangle$. We then have the infinite series

$$|\psi\rangle = |\phi\rangle + \frac{1}{E-H_0+i\epsilon}V|\phi\rangle + \frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V|\phi\rangle$$

$$+\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V|\phi\rangle+\cdots. \tag{32}$$

This is called Born expansion, and the Born approximation we used is nothing but the first term in this systematic expansion. The physical meaning of this equation is obvious. The first term is the wave which did not get scattered. The second term is the wave that gets scattered at a point in the potential and then propagates outwards by the $1/(E-H_0+i\epsilon)$ operator. In the third term, the wave gets scattered at a point in the potential, propagates for a while, and gets scattered again at another point in the potential, and propagates outwards. In the $n+1$-th term, there are n times scattering of the wave before it propagates outwards.

More formally, an operator called T-matrix is used often in scattering problems. The definition is

$$V|\psi\rangle = T|\phi\rangle. \tag{33}$$

We always take $|\phi\rangle = |\hbar\vec{k}\rangle$. This seemingly weird definition is actually useful as seen below. The scattering amplitude derived in the lecture note "Scattering Theory I" is

$$f(\vec{k'},\vec{k}) = -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\langle\hbar\vec{k'}|V|\psi\rangle. \tag{34}$$

Using the definition of the T-matrix, we find

$$f(\vec{k'},\vec{k}) = -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\langle\hbar\vec{k'}|T|\hbar\vec{k}\rangle. \tag{35}$$

Hence, the T-matrix element has a physical interpretation of the transition (hence T) from the initial momentum $\hbar\vec{k}$ to the final momentum $\hbar\vec{k'}$.

Using the Lippmann–Schwinger equation Eq. (1), and multiplying the both sides by V from left, we find

$$T|\phi\rangle = V|\phi\rangle + V\frac{1}{E-H_0+i\epsilon}T|\phi\rangle, \tag{36}$$

and hence

$$T = V + V\frac{1}{E-H_0+i\epsilon}T. \tag{37}$$

In other words, a formal solution to the T-matrix is

$$T = \frac{1}{1-V\frac{1}{E-H_0+i\epsilon}}V. \tag{38}$$

By Taylor expanding this operator in geometric series, we find

$$T = V + V\frac{1}{E - H_0 + i\epsilon}V + V\frac{1}{E - H_0 + i\epsilon}V\frac{1}{E - H_0 + i\epsilon}V + \cdots. \quad (39)$$

This proves the Born expansion Eq. (32).

In the coordinate space, for example, the second Born term is given by

$$\begin{aligned} &\langle\vec{x}|\frac{1}{E - H_0 + i\epsilon}V\frac{1}{E - H_0 + i\epsilon}V|\phi\rangle \\ &= \int d\vec{x'}d\vec{x''}\frac{-2m}{\hbar^2}\frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|}V(\vec{x'})\frac{-2m}{\hbar^2}\frac{e^{ik|\vec{x'}-\vec{x''}|}}{4\pi|\vec{x'}-\vec{x''}|}V(\vec{x''})\phi(\vec{x''}), \quad (40) \end{aligned}$$

where $\phi(\vec{x''}) = e^{i\vec{k}\cdot\vec{x''}}/(2\pi\hbar)^{3/2}$.

4 Validity of Born Approximation

Born approximation replaces ψ by ϕ in Lippmann–Schwinger equation, which is integrated together with the potential. Therefore, in order for Born approximation to be good, the difference between ψ and ϕ must be small where the potential exists. The self-consistency requires that

$$|\psi(\vec{x}) - \phi(\vec{x})| \ll |\phi(\vec{x})| \quad (41)$$

where $V(\vec{x})$ is sizable, and the l.h.s. can be evaluated within Born approximation itself. From Lippmann–Schwinger equation (the one before taking the limit of large r), we find

$$\left|\frac{2m}{\hbar^2}\int d\vec{x'}\frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|}V(\vec{x'})e^{i\vec{k}\cdot\vec{x'}}\right| \ll 1. \quad (42)$$

In particular, we require this condition at $\vec{x} = 0$ where the potential is the strongest presumably.

For a smooth central potential, with a magnitude of order V_0 and a range of order a, we can qualitatively work out the validity constraint Eq. (42). Taking $\vec{k}$ along the z axis, and looking at $\vec{x} \simeq 0$ where the potential is most important presumably (and relabeling $\vec{x'}$ as $\vec{x}$), the condition is

$$\frac{2m}{\hbar^2}\left|\int d\vec{x}\frac{e^{ikr}}{4\pi r}V(\vec{x})e^{ikz}\right| \ll 1. \quad (43)$$

When $k \ll a^{-1}$, we can ignore the phases in the integral, and it is given roughly by

$$\frac{2m}{\hbar^2}|V_0|a^2\frac{1}{2} \ll 1 \qquad (k \ll a^{-1}). \tag{44}$$

Numerical coefficients are not to be trusted. On the other hand, when $k \gg a^{-1}$, the phase factor oscillates rapidly and we can use stationary phase approximation. The exponent is $ikr + ikz$, and it is stationary only along the negative z-axis $z = -r$. Expanding around this point, it is $ikr + ikz = ik(x^2 + y^2)/r + O(x^3, y^3)$. The Gaussian integral over x, y then gives a factor of $\pi r/k$, while z is integrated along the stationary phase direction from $-a$ to 0. Therefore, the validity condition is given roughly by

$$\frac{2m}{\hbar^2}\frac{a}{4k}|V_0| \ll 1 \qquad (k \gg a^{-1}). \tag{45}$$

On the other hand, we can estimate the total cross section in both limits. The scattering amplitude in the Born approximation Eq. (5) is

$$\begin{aligned} f^{(1)}(\vec{k'}, \vec{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\vec{x} V(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \\ &\sim -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\frac{4\pi}{3}a^3 \qquad (q \ll a^{-1}). \end{aligned} \tag{46}$$

For a large momentum transfer, say along the x axis, y and z integral each gives a factor of a because of no phase variation, while x integral oscillates rapidly and cancels mostly; it leaves only $\sim 1/q$ contribution from non-precise cancellation. Therefore,

$$f^{(1)}(\vec{k'}, \vec{k}) \sim -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\frac{\pi a^2}{q} \qquad (q \gg a^{-1}). \tag{47}$$

Because the momentum transfer q is of the order of k (except the very forward region which we neglect from this discussion), the total cross sections are roughly

$$\sigma \sim \left\{ \begin{array}{ll} \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}V_0\frac{4\pi}{3}a^3\right)^2 & (k \ll a^{-1}) \\ \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}V_0\frac{\pi a^2}{q}\right)^2 & k \gg a^{-1}). \end{array} \right. \tag{48}$$

It is interesting to note that, once the validity condition Eqs. (44,45) is satisfied, the total cross section is always smaller than the geometric cross section $4\pi a^2$.

$$\sigma \ll \frac{16}{9}\pi a^2 \qquad (k \ll a^{-1}) \tag{49}$$

$$\sigma \ll 4\pi a^2 \qquad (k \gg a^{-1}). \tag{50}$$

If you find a Born cross section larger than the geometric cross section, you should be worried.