

Final Exam Solutions

1. Pair production of electrons from two photons.

- (a) I refer to the initial four-momentum of the cosmic ray photon by q_1^μ and the photon in the background q_2^μ . The requirement for the pair production is $(q_1 + q_2)^2 = 2q_1 \cdot q_2 > 4m_e^2$. We choose our coordinate system such that q_1 is along the z -axis, $q_1^\mu = E_1(1, 0, 0, 1)$. According to the assumption in the problem, photons in the background have four-momenta $q_2^\mu = E_2(1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ with $E_2 = 3 \times 10^{-4}$ eV. $2q_1 \cdot q_2 = 2E_1E_2(1 - \cos \theta)$, which is maximized when $\cos \theta = -1$. Therefore the minimum energy is determined by $(2q_1 \cdot q_2)_{\max} = 4E_1E_2 > 4m_e^2$, *i.e.*, $E_1 > m_e^2/E_2 = (0.511 \text{ MeV})^2/(3 \times 10^{-4} \text{ eV}) = 8.70 \times 10^5 \text{ GeV} = 870 \text{ TeV}$.
- (b) I believe these are obvious.
- (c) You always have the following common factors. The amplitude has two coupling constants e^2 , the phase space $\frac{\beta_f}{8\pi}$ and the flux factor $\frac{1}{2s}$. The kinematical factors in the amplitude are different between two cases. (i) Close to the threshold, the wave functions of electron, positron are roughly $\sim \sqrt{m}$, and the t - or u - channel electron propagators $(\not{p} + m)/(p^2 - m^2) \sim 1/m$. The photon polarization vectors are $O(1)$. Therefore, the amplitude behaves as $\mathcal{M} \sim e^2$. The cross section then is $\sigma \sim \frac{\beta_f}{8\pi} \frac{1}{8m^2} (e^2)^2 = \frac{\pi\alpha^2\beta_f}{4m^2}$. (ii) Well above the threshold, the electron, positron spinors are $\sim \sqrt{2E}$, while the electron propagator $\sim 1/E$. Therefore the amplitude is roughly $\mathcal{M} \sim 2e^2$. There is, however, an angular dependence of the amplitude, which is actually $\mathcal{M} \sim 2e^2 \sin(\theta/2)/(1 - \cos \theta)$ or $2e^2 \cos(\theta/2)/(1 + \cos \theta)$ which gives us a logarithmic enhancement factor. We'll put this back in at the very end. Setting the angular dependence aside, we obtain $\sigma \sim \frac{1}{8\pi} \frac{1}{2s} (2e^2)^2 = \frac{4\pi\alpha^2}{s}$. Now we put the logarithmic enhancement factor in, and the estimate is $\sigma \sim \frac{4\pi\alpha^2}{s} \log \frac{s}{m^2}$.
- (d) The amplitude is

$$\begin{aligned}
 i\mathcal{M} = & (-ie)^2 \left(\bar{u}(p)\gamma^\mu \frac{i}{\not{k}_1 - m} \gamma^\nu v(\bar{p})\epsilon_\mu(q_1)\epsilon_\nu(q_2) \right. \\
 & \left. + \bar{u}(p)\gamma^\mu \frac{i}{\not{k}_2 - m} \gamma^\nu v(\bar{p})\epsilon_\mu(q_2)\epsilon_\nu(q_1) \right) \quad (1)
 \end{aligned}$$

Here, $k_1 = p - q_1$ and $k_2 = p - q_2$. The relative sign between two amplitudes is plus, because they are related by Bose symmetry, $q_1 \leftrightarrow q_2$.

- (e) We calculated the spin-summed squared amplitude for the Compton scattering in the class. For initial four-momenta p_i (k_i) of electron (photon), and similarly for final ones, we found

$$\begin{aligned} & \sum_{\text{helicities}} |\mathcal{M}|^2 \\ &= 8e^4 \left\{ \frac{p_i \cdot k_f}{p_i \cdot k_i} + \frac{p_i \cdot k_i}{p_i \cdot k_f} + 2m^2 \left(\frac{1}{p_i \cdot k_i} - \frac{1}{p_i \cdot k_f} \right) + m^4 \left(\frac{1}{p_i \cdot k_i} - \frac{1}{p_i \cdot k_f} \right)^2 \right\} (2) \end{aligned}$$

Using crossing, we can obtain the spin-summed squared amplitude for $\gamma\gamma \rightarrow e^+e^-$ by substituting $k_i \rightarrow q_1$, $k_f \rightarrow -q_2$, $p_i \rightarrow -\bar{p}$, $p_f \rightarrow p$, and changing the overall sign.

$$\begin{aligned} & \sum_{\text{helicities}} |\mathcal{M}|^2 \\ &= 8e^4 \left\{ \frac{\bar{p} \cdot q_2}{\bar{p} \cdot q_1} + \frac{\bar{p} \cdot q_1}{\bar{p} \cdot q_2} + 2m^2 \left(\frac{1}{\bar{p} \cdot q_1} + \frac{1}{\bar{p} \cdot q_2} \right) - m^4 \left(\frac{1}{\bar{p} \cdot q_1} + \frac{1}{\bar{p} \cdot q_2} \right)^2 \right\} . (3) \end{aligned}$$

The symmetry between $q_1 \leftrightarrow q_2$ in the result is apparent. What is less apparent is the symmetry $p \leftrightarrow \bar{p}$: $2\bar{p} \cdot q_2 = \bar{p}^2 + q_2^2 - (\bar{p} - q_2)^2 = m^2 + 0 - (q_1 - p)^2 = 2p \cdot q_1$. This symmetry holds thanks to the charge conjugation invariance of the QED which interchanges electron (momentum p) and positron (momentum \bar{p}) in the final states, but does not affect the initial state.

- (f) The total cross section is given by

$$\sigma = \left(\frac{1}{2} \right)^2 \frac{\beta_f}{8\pi} \frac{1}{2s} \int \frac{d\Omega}{4\pi} \sum_{\text{helicities}} |\mathcal{M}|^2 \quad (4)$$

First, one needs to write out the inner products of four-momenta explicitly. Let us use the center-of-momentum frame to simplify it. (Remember the total cross section is Lorentz-invariant.) Use $q_1^\mu = E(1, 0, 0, 1)$, $q_2^\mu = E(1, 0, 0, -1)$, $p^\mu = (E, p \sin \theta, 0, p \cos \theta)$, $\bar{p}^\mu = (E, -p \sin \theta, 0, -p \cos \theta)$, with $p = E\beta = \sqrt{E^2 - m^2}$. Here, we have chosen the rotational invariance to fix the p direction to lie in the xz plane. Then, $\bar{p} \cdot q_1 = E^2 + Ep \cos \theta$, $\bar{p} \cdot q_2 = E^2 - Ep \cos \theta$. The spin-summed

squared amplitude becomes

$$\begin{aligned} \sum_{\text{helicities}} |\mathcal{M}|^2 &= 8e^4 \left\{ \frac{E^2 - Ep \cos \theta}{E^2 + Ep \cos \theta} + \frac{E^2 + Ep \cos \theta}{E^2 - Ep \cos \theta} \right. \\ &\quad \left. + 2m^2 \left(\frac{1}{E^2 + Ep \cos \theta} + \frac{1}{E^2 - Ep \cos \theta} \right) \right. \\ &\quad \left. - m^4 \left(\frac{1}{E^2 + Ep \cos \theta} + \frac{1}{E^2 - Ep \cos \theta} \right)^2 \right\}. \end{aligned} \quad (5)$$

An integral of the terms inside the curly brackets over $\cos \theta = -1$ to 1 gives

$$\frac{-4 E^3 p - 4 E m^2 p + (8 E^4 + 8 E^2 m^2 - 4 m^4) \operatorname{arctanh}\left(\frac{p}{E}\right)}{E^3 p} \quad (6)$$

(Hey, I obtained a lot messier expression before; this is a lot simpler!) Therefore,

$$\begin{aligned} \sigma &= \left(\frac{1}{2}\right)^2 \frac{\beta_f}{8\pi} \frac{1}{2s} \int \frac{d \cos \theta}{2} \sum_{\text{helicities}} |\mathcal{M}|^2 \\ &= \frac{1}{4} \frac{\beta_f}{8\pi} \frac{1}{2s} 8e^4 \frac{1}{2} \left\{ \frac{-4 E^3 p - 4 E m^2 p + (8 E^4 + 8 E^2 m^2 - 4 m^4) \operatorname{arctanh}\left(\frac{p}{E}\right)}{E^3 p} \right\} \\ &= \frac{e^4 \beta_f}{16\pi s} \left\{ \frac{-4 E^3 p - 4 E m^2 p + (8 E^4 + 8 E^2 m^2 - 4 m^4) \operatorname{arctanh}\left(\frac{p}{E}\right)}{E^3 p} \right\} \end{aligned} \quad (7)$$

We show a plot of the cross section in Fig. 1.

(g) The terms in the curly brackets have a limit 4 when $p \rightarrow 0$, $E \rightarrow m$. Therefore,

$$\sigma = \frac{e^4 \beta_f}{16\pi s} 4 = \frac{\pi \alpha^2 \beta_f}{m^2}. \quad (8)$$

The estimate in (b) was a factor of four off. Not too bad.

(h) Let us use an approximate order of magnitude for the cross section, $\pi \alpha^2 / m^2$. The mean free path due to a scattering with the background photons is roughly

$$l^{-1} \sim \frac{\pi \alpha^2}{m^2} 2 \frac{\zeta(3)}{\pi^2} T_0^3 = 2.0 \times 10^{-33} \text{ MeV} \quad (9)$$

and using $\hbar c = 197 \text{ MeV fm}$ (this is a useful constant to remember!)

$$l \sim 9.7 \times 10^{21} \text{ cm} = 3.15 \text{ kpc} \quad (10)$$

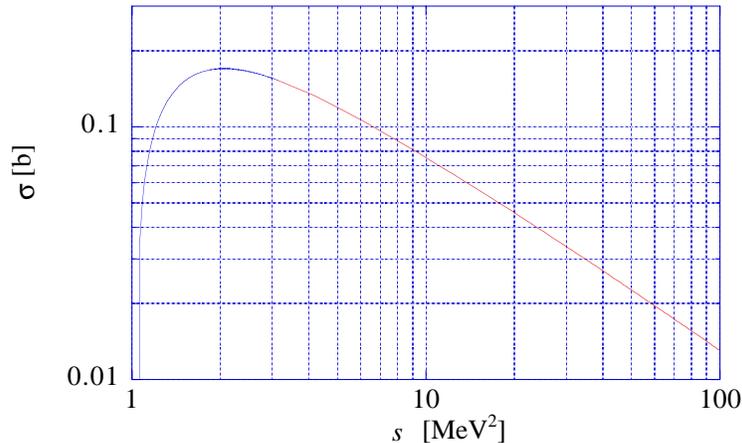


Figure 1: The cross section of $\gamma\gamma \rightarrow e^+e^-$ in the unit of barn $1 \text{ b} = 10^{-24} \text{ cm}^2$, as a function of $s = E_{CM}^2$.

This is somewhat shorter than the distance between the Earth and our galactic center $\sim 8 \text{ kpc}$, but we cannot say it is definitely shorter because we made only a crude estimate. To be more precise, we need to integrate over the Planckian distribution of photon energies (and also directions). However, this is not important anyway because there are many other and more important processes which can scatter incoming photons inside the galactic disk. The background photons are important when there is basically “nothing” between the source and the Earth, which is the case for an extragalactic source of high energy photons.¹ The calculated mean free path is much shorter than an intergalactic distance. Therefore, ultra-high-energy gamma rays from extragalactic sources are all converted to electron-positron pairs due to scattering with microwave background photons. Their fate is to cause showers due to electromagnetic cascades. The produced electron-positron pairs further lose energies by emitting bremsstrahlung photons, or cause further pair production by scattering with microwave background photons, and result in a larger number of electrons, positrons, and photons in a narrow cone. They are called “showers.” Such high energy photons, therefore, can still be detected in a form of showers even though the primary photon itself won’t reach the Earth directly. But this tells us that looking for photons (traditional astronomy) wouldn’t be so useful at very high energies.

¹To the best of my knowledge, whether there exist infrared radiations which may potentially scatter the high-energy photons is not understood.

Note Many of you asked what the heck $\zeta(3)$ is. This is a very standard outcome in statistical mechanics of relativistic particles, but maybe you are not familiar with it. The number density of a boson is given by

$$n = g \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \frac{1}{e^{\beta(E-\mu)} - 1} \quad (11)$$

As usual, $\beta = 1/kT$. The factor $g = 2$ counts the spin degrees of freedom. Photons do not have a chemical potential, and $E = cp$. Therefore,

$$n = g \int \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{1}{e^{\beta cp} - 1} \quad (12)$$

Now you expand the integrand in the following manner,

$$n = g \frac{1}{2\pi^2\hbar^3} \int_0^\infty p^2 dp \sum_{n=1}^\infty e^{-n\beta cp} \quad (13)$$

Integrating each terms in the sum gives the Γ -function, and we obtain

$$n = g \frac{1}{2\pi^2\hbar^3} \Gamma(3) \sum_{n=1}^\infty \frac{1}{(n\beta c)^3} = g \frac{(kT)^3}{\pi^2(\hbar c)^3} \sum_{n=1}^\infty \frac{1}{n^3} \quad (14)$$

The last factor is $\zeta(3) = 1.20206\dots$ which is an irrational number like e or π . In general, $\zeta(s) \equiv \sum_{n=1}^\infty n^{-s}$ Finally, we have

$$n = g \frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c} \right)^3 \quad (15)$$

I have further taken the natural unit, $k = \hbar = c = 1$ in the problem.

If you have a thermal gas of relativistic fermions, the only change in the above calculation is that you have a sum over series with alternating signs,

$$\frac{1}{e^{\beta cp} + 1} = \sum_{n=1}^\infty (-1)^{n-1} e^{-n\beta cp} \quad (16)$$

The result of the phase space integral also has a sum over alternating signs,

$$n = g \frac{1}{\pi^2(\hbar c)^3} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3} \quad (17)$$

The summation can be done using the following trick.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} - 2 \left(\frac{1}{2^3} + \frac{1}{4^3} + \dots \right) = \zeta(3) - 2 \frac{1}{2^3} \zeta(3) = \frac{3}{4} \zeta(3) \quad (18)$$

Therefore,

$$n = g \frac{3 \zeta(3)}{4 \pi^2} \left(\frac{kT}{\hbar c} \right)^3 \quad (19)$$

for relativistic fermions. You can also calculate the energy density in a similar way. It helps to know that $\zeta(2) = \pi^2/6$. You obtain Stefan–Boltzman law of the energy density of the black body radiation in this manner.

2. beta-function in ϕ^4 theory.

- (a) The Feynman rule is that we put a factor of $-i\lambda$ for each four-point scalar vertex. The one-loop amplitude is

$$-i\frac{\lambda}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \quad (20)$$

The reason for $1/2$ is the following. From the definition of the two-point function in the interaction picture, the term at $O(\lambda)$ is given by the correlation function,

$$\langle 0|T\phi(x)\phi(y) \int d^4z (-i) \frac{\lambda}{4!} \phi(z)^4 |0\rangle \quad (21)$$

There are four choices on which $\phi(z)$ is contracted with $\phi(x)$, and three more choices on which of the remaining three $\phi(z)$ is contracted with $\phi(y)$. There still remains two $\phi(z)$'s, which are contracted with each other to give the above one-loop amplitude. Therefore, the overall factor is

$$-i\frac{\lambda}{4!} \times 4 \times 3 = -i\frac{\lambda}{2} \quad (22)$$

It has an additional factor of $1/2$ compared to a naive expectation. In general, a loop amplitude which involves a real scalar by itself has an additional multiplicative factor of $1/2$.

The result does not depend on the four-momentum of the external line. Therefore, it cannot contribute to the coefficient of q^2 term in the 1PI two-point function, and hence not to the wave-function renormalization. The wave function renormalization appears only at the two-loop level in this theory. On the other hand, the above one-loop amplitude renormalizes the mass of the scalar boson. From the power counting, it is easy to see that it is *quadratically* divergent! This makes ϕ^4 theory very sensitive to physics at the ultraviolet cutoff. Not very nice. This is called the “naturalness problem” and is relevant to the Higgs boson in the standard model.

- (b) Call the initial four-momenta to be p_1 and p_2 , and final ones p_3 and p_4 . The one-loop amplitude then is given by

$$i\mathcal{M} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(4\pi)^2} \frac{i}{(q - p_1 - p_2)^2 - m^2} \frac{i}{q^2 - m^2} \\ + ((p_1 + p_2) \rightarrow (p_1 - p_3)) + ((p_1 + p_2) \rightarrow (p_1 - p_4)) \quad (23)$$

Here again we have a multiplicative factor of $1/2$. The reason is the same as the above case. We need to calculate the four-point correlation function at $O(\lambda^2)$. It comes from the following correlation function,

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\frac{1}{2!}\int d^4z_1(-i)\frac{\lambda}{4!}\phi(z_1)^4\int d^4z_2(-i)\frac{\lambda}{4!}\phi(z_2)^4|0\rangle \quad (24)$$

Here, the reason for $1/2!$ is because it is the second-order term in the Taylor expansion of $T e^{-i\int d^4z\mathcal{H}_{\text{int}}}$. But this is not the reason for the final $1/2$. First of all, there are three different types of Wick contractions. One is to contract $\phi(x_1)$ and $\phi(x_2)$ with the same z , say $\phi(z_1)^4$, and $\phi(x_3)$ and $\phi(x_4)$ with $\phi(z_2)^4$ (s -channel). The other is to contract $\phi(x_1)$ and $\phi(x_3)$ with the same z (t -channel), the last $\phi(x_1)$ and $\phi(x_4)$ with the same z (u -channel). This is why we have a sum of three different terms. Now let us look at one particular contraction, where $\phi(x_1)$ and $\phi(x_2)$ are contracted with the same z (s -channel diagram). First of all, there are two choices, whether they are contracted with z_1 or z_2 . Both of them give the same amplitude, and they are added together. This factor of two cancels the factor of $1/2!$ in the Taylor expansion. We pick z_1 and drop the factor of $1/2!$ hereafter. Now there are four choices on which $\phi(z_1)$ to contract with $\phi(x_1)$, and three more on which remaining three $\phi(z_1)$ to contract with $\phi(x_2)$. At this point, we have a factor of $-i\lambda/4!*4*3 = -i\lambda/2$. Exactly the same situation occurs with the contraction of $\phi(x_3)$, $\phi(x_4)$ with $\phi(z_2)^4$. At this point, two $\phi(z_1)^2$ and $\phi(z_2)^2$ remain uncontracted. They are contracted with each other, with two possibilities. Therefore, the overall coupling constant factor is $(-i\lambda/2)^2 * 2 = (-i\lambda)^2/2$. You can check that exactly the same factor appears also for t -channel and u -channel amplitudes.

(c) After setting all external momenta to vanish, we obtain

$$i\mathcal{M} = \frac{3}{2}\lambda^2 \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p^2 - m^2} \right)^2 \quad (25)$$

because s -, t -, and u -channel amplitudes become the same. One way to evaluate the amplitude is to use Pauli-Villars regulator with a mass M , and do the following.

$$\begin{aligned} i\mathcal{M} &= \frac{3}{2}\lambda^2 \int \frac{d^4p}{(2\pi)^4} \left[\left(\frac{1}{p^2 - m^2} \right)^2 - \left(\frac{1}{p^2 - M^2} \right)^2 \right] \\ &= \frac{3}{2}\lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_{m^2}^{M^2} d\mu^2 (-2) \left(\frac{1}{p^2 - \mu^2} \right)^3 \end{aligned} \quad (26)$$

Now we perform the p integration first using the by-now-familiar formula,

$$\begin{aligned}
i\mathcal{M} &= \frac{3}{2}\lambda^2 \int_{m^2}^{M^2} d\mu^2 (-2) \frac{(-1)^3 i \Gamma(1)}{(4\pi)^2 \Gamma(3)} \frac{1}{\mu^2} \\
&= \frac{3}{2}\lambda^2 (-2) \frac{-i \Gamma(1)}{(4\pi)^2 \Gamma(3)} \log \frac{M^2}{m^2} \\
&= i \frac{3}{2} \frac{\lambda^2}{(4\pi)^2} \log \frac{M^2}{m^2}
\end{aligned} \tag{27}$$

Another method is an explicit cutoff. Going back to Eq. (25), we first perform the Wick rotation and cutoff the momentum integration with $p_E^2 < M^2$,

$$i\mathcal{M} = \frac{3}{2}\lambda^2 i \int \frac{d^4 p_E}{(2\pi)^4} \left(\frac{1}{p_E^2 + m^2} \right)^2 \tag{28}$$

$$= i \frac{3}{2} \lambda^2 \int_0^{M^2} \frac{\pi^2 p_E^2 dp_E^2}{(2\pi)^4} \left(\frac{1}{p_E^2 + m^2} \right)^2 \tag{29}$$

The integration can be done in parts,

$$\begin{aligned}
i\mathcal{M} &= i \frac{3}{2} \lambda^2 \frac{1}{(4\pi)^2} \left(p_E^2 \frac{-1}{p_E^2 + m^2} \Big|_0^{M^2} - \int_0^{M^2} dp_E^2 \frac{-1}{p_E^2 + m^2} \right) \\
&= i \frac{3}{2} \lambda^2 \frac{1}{(4\pi)^2} \left(-1 + \log(p_E^2 + m^2) \Big|_0^\infty \right) \\
&= i \frac{3}{2} \lambda^2 \frac{1}{(4\pi)^2} \left(-1 + \log \frac{M^2}{m^2} \right)
\end{aligned} \tag{30}$$

The results based on two different methods disagree on the constant term, but the logarithmically divergent piece have the same coefficient.

- (d) The calculated $i\mathcal{M}$ above is added to the bare coupling $-i\lambda_0$ at the lowest order, and hence,

$$\lambda = \lambda_0 - \frac{3}{2} \frac{\lambda_0^2}{(4\pi)^2} \log \frac{M^2}{m^2} \tag{31}$$

According to the definition of the β -function in the problem,

$$\beta(\lambda) = -M^2 \frac{d\lambda}{dM^2} \Big|_{\lambda_0} = \frac{3}{2} \frac{\lambda^2}{(4\pi)^2} \tag{32}$$

Here, we used the fact that the λ_0 in the one-loop term can be replaced by λ up to higher order corrections.

Note In this problem, a couple of you asked me what is the momentum scale Q^2 here analogous to that in the effective QED coupling constant. The answer is somewhat complicated. The amplitude here depends on three kinematical variables, s , t , and u . This is actually the one-particle-irreducible (1PI) four-point function $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$, or equivalently, $\Gamma^{(4)}(s, t, u)$. How you define an effective coupling constant as a function of a *single* momentum scale Q^2 is up to you. One choice made in the textbook is $s = t = u = -Q^2$, which is in the *unphysical* region, *i.e.*, such a kinematics never arises for an S -matrix element. But this is a popular definition. Another choice is to use dimensional regularization and the so-called $\overline{\text{MS}}$ renormalization scheme, which is also popular.

Whatever your definition is, the following is always true, which does not require you to define the effective coupling constant as a function of a single kinematical variable. It is the repetition of the argument we did in the class. The 1PI four-point function is a function of external momenta, the bare coupling constant, and the ultraviolet cutoff parameter M . To be more explicit, $\Gamma^{(4)} = \Gamma^{(4)}(s, t, u, \lambda_0, M^2)$. Now we study the same function with a *scaling* of the four-momenta, $\Gamma^{(4)}(e^\alpha s, e^\alpha t, e^\alpha u, \lambda_0, M^2)$. In other words, we study the same 1PI four-point function with all kinematical variables scaled by the same factor e^α . If we neglect the mass m , a dimensional analysis tells you that $\Gamma^{(4)}(e^\alpha s, e^\alpha t, e^\alpha u, \lambda_0, M^2) = \Gamma^{(4)}(s, t, u, \lambda_0, e^{-\alpha} M^2)$. Therefore,

$$\begin{aligned} \left(s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} \right) \Gamma^{(4)}(s, t, u, \lambda_0, M^2) &= \frac{\partial}{\partial \alpha} \Gamma^{(4)}(e^\alpha s, e^\alpha t, e^\alpha u, \lambda_0, M^2) \Big|_{\alpha=0} \\ &= \frac{\partial}{\partial \alpha} \Gamma^{(4)}(s, t, u, \lambda_0, e^{-\alpha} M^2) \Big|_{\alpha=0} \\ &= -M^2 \frac{\partial}{\partial M^2} \Gamma^{(4)}(s, t, u, \lambda_0, M^2) \quad (33) \end{aligned}$$

And (at least at the one-loop level) we find the last expression to be independent of the kinematics s , t , or u . Therefore, what you have calculated is useful independent of the precise definition of the effective coupling constant $\lambda(Q^2)$.

- (e) (I am aware that this question was confusing.) In dimensional regularization method, the bare coupling constant λ_0 has a mass dimension of 2ϵ , where I take the space-time dimension to be $D = 4 - 2\epsilon$. The one-loop amplitude in Eq. (25) is now

$$i\mathcal{M}\mu^{2\epsilon} = \frac{3}{2}\lambda_0^2 \int \frac{d^D p}{(2\pi)^D} \left(\frac{1}{p^2 - m^2} \right)^2 = \frac{3}{2}\lambda_0^2 \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(2)} (m^2)^{-\epsilon} \quad (34)$$

using the formulae which we discussed in the class. The factor $\mu^{2\epsilon}$ is introduced to make the amplitude \mathcal{M} dimensionless, and μ is an arbitrary parameter with unit mass dimension. By differentiating the total amplitude (with tree-level term)

$$\lambda = -\mathcal{M} = \lambda_0 \mu^{-2\epsilon} - \frac{3}{2} \frac{\lambda_0^2}{(4\pi)^{2-\epsilon}} \mu^{-2\epsilon} \Gamma(\epsilon) (m^2)^{-\epsilon} \quad (35)$$

by μ^2 with λ_0 fixed,

$$\beta = \mu^2 \frac{d\lambda}{d\mu^2} = \lambda_0 (-\epsilon) \mu^{-2\epsilon} - \frac{3}{2} \frac{\lambda_0^2}{(4\pi)^{2-\epsilon}} (-\epsilon) \mu^{-2\epsilon} \Gamma(\epsilon) (m^2)^{-\epsilon} \rightarrow \frac{3}{2} \frac{\lambda^2}{(4\pi)^2} \quad (36)$$

In the last part, we took the limit $\epsilon \rightarrow 0$, and used a perturbative expansion $\lambda_0 = \lambda \mu^{2\epsilon} + O(\lambda^2)$.

Note Usually we expand the amplitude in power series in ϵ ,

$$\begin{aligned} \lambda = -\mathcal{M} &= \lambda_0 \mu^{-2\epsilon} - \frac{3}{2} \frac{(\lambda_0 \mu^{-2\epsilon})^2}{(4\pi)^{2-\epsilon}} \left(\frac{1}{\epsilon} - \gamma + O(\epsilon) \right) \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \\ &\lambda_0 \mu^{-2\epsilon} - \frac{3}{2} \frac{(\lambda_0 \mu^{-2\epsilon})^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log(4\pi) - \log \frac{m^2}{\mu^2} + O(\epsilon) \right) \end{aligned} \quad (37)$$

If you define the renormalized coupling constant $\lambda(\mu^2)$ by absorbing the pole and $-\gamma + \log(4\pi)$ pieces,

$$\lambda(\mu^2) = \lambda_0 \mu^{-2\epsilon} - \frac{3}{2} \frac{(\lambda_0 \mu^{-2\epsilon})^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log(4\pi) \right) \quad (38)$$

it is called the $\overline{\text{MS}}$ coupling constant. This scheme is very commonly used in QCD.

Note This exercise is a “sloppy” way of calculating the β -function in dimensional regularization, which works only at one-loop. I learnt this method from the book by Ryder. Let me describe what the “correct” method is. As usual, we focus on the 1PI four-point function, $\Gamma^{(4)}(s, t, u, \lambda_0)$ which is calculated with the bare coupling λ_0 . Note that in the bare theory, $\Gamma^{(4)}$ has a mass dimension of 2ϵ . What we are interested is the momentum dependence of the four-point function. We scale all four-momenta by an overall common factor e^α , and use the dimensional analysis,

$$\Gamma^{(4)}(e^\alpha s, e^\alpha t, e^\alpha u, \lambda_0) = e^{\epsilon\alpha} \Gamma^{(4)}(s, t, u, e^{-\epsilon\alpha} \lambda_0) \quad (39)$$

Now we define the “renormalized” four-point function by

$$\Gamma_R^{(4)}(s, t, u, \lambda) = \mu^{-2\epsilon} (Z^{-1/2})^4 \Gamma^{(4)}(s, t, u, \lambda_0) \quad (40)$$

which is dimension-less. Note that we regard the renormalized four-point function as a function of λ , not λ_0 . Z is the wave-function renormalization factor. In our case, it is unity at the one-loop level. Using the definition of Γ_R and also the dimensional analysis of Γ , we find

$$\Gamma_R^{(4)}(e^\alpha s, e^\alpha t, e^\alpha u, \lambda, \mu^2) = \Gamma_R^{(4)}(s, t, u, \lambda, e^{-\alpha} \mu^2) \quad (41)$$

Therefore, the “correct” way is to fix λ , differentiate it with μ^2 and *change the sign*. However, it is equivalent to fixing λ_0 and not changing the sign because the amplitude is

$$\lambda_0^2 \mu^{-2\epsilon} (m^2)^{-\epsilon} = \lambda^2 \mu^{2\epsilon} (m^2)^{-\epsilon} \quad (42)$$

at this order.

(f) By writing down the differential equation,

$$Q^2 \frac{d\lambda}{dQ^2} = \frac{3}{2} \frac{1}{(4\pi)^2} \lambda^2 \quad (43)$$

in a slightly different form,

$$\frac{d\lambda}{\lambda^2} = \frac{3}{2} \frac{1}{(4\pi)^2} d \log Q^2, \quad (44)$$

both sides can be easily integrated and we obtain

$$-\frac{1}{\lambda(Q_1^2)} + \frac{1}{\lambda(Q_2^2)} = \frac{3}{2} \frac{1}{(4\pi)^2} \log \frac{Q_1^2}{Q_2^2}, \quad (45)$$

or,

$$\lambda(Q_2^2) = \frac{\lambda(Q_1^2)}{1 - \frac{3}{2} \frac{\lambda(Q_1^2)}{(4\pi)^2} \log \frac{Q_2^2}{Q_1^2}} \quad (46)$$

Therefore, the coupling constant grows for larger energies, or decreases for smaller energies: “infrared-free.” The situation is very similar to that in the QED. However, if we have such a theory with a relatively large size of λ at energy scale directly observable by experiments, the coupling constant may diverge not too far above such energy scales. If you require, for instance, that the coupling constant

stays finite up to the Planck scale, you obtain an upper bound on the size of $\lambda(m_Z^2)$, for instance.

Actually, the Higgs boson in the Standard Model has ϕ^4 -type coupling, and the mass squared of the Higgs boson is proportional to coupling constant λ . Therefore, an upper bound on λ translates into an upper bound on the Higgs boson mass. By requiring that the coupling constant remains perturbative up to the Planck scale, we obtain $m_H < 150$ GeV or so.

3. Electron Electric Dipole Moment.

The main point of this calculation is that the contribution vanishes if only ϕ_L (or ϕ_R) propagates inside the vertex diagram because it would give a combination

$$\bar{u}(p') \frac{1 + \gamma_5}{2} \dots \frac{1 - \gamma_5}{2} u(p), \quad (47)$$

and even if $i\sigma^{\mu\nu}q_\nu$ is obtained inside \dots , it commutes with γ_5 and the end result vanishes because $\frac{1+\gamma_5}{2}\frac{1-\gamma_5}{2} = 0$. Therefore, we need a diagram which starts with ϕ_L and ends with ϕ_R or vice versa. This is why we need to use “vertex” $-iAm$ or $-iA^*m$ in the diagram. If the diagram starts with ϕ_L and ends with ϕ_R , the structure of the amplitude becomes

$$\bar{u}(p') \frac{1 - \gamma_5}{2} \dots \frac{1 - \gamma_5}{2} u(p), \quad (48)$$

and $i\sigma^{\mu\nu}q_\nu$ term in \dots would survive. Given this point, and using the Feynman rule for the scalar QED given in the book (p. 312), we find the following amplitude when ϕ_L is emitted from the electron, converts to ϕ_R on the way and is absorbed by the electron:

$$\begin{aligned} i\mathcal{M}_{LR} = & \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-i\sqrt{2}e) \frac{1 - \gamma_5}{2} \frac{i}{\not{k} - \mu} (-i\sqrt{2}e) \frac{1 - \gamma_5}{2} u(p) \\ & \left[\frac{i}{(p' - k)^2 - M_R^2} (-iA^*m) \frac{i}{(p' - k)^2 - M_L^2} (-ie)(p + p' - 2k)^\mu \frac{i}{(p - k)^2 - M_L^2} \right. \\ & \left. + \frac{i}{(p' - k)^2 - M_R^2} (-ie)(p + p' - 2k)^\mu \frac{i}{(p' - k)^2 - M_R^2} (-iA^*m) \frac{i}{(p - k)^2 - M_L^2} \right]. \end{aligned} \quad (49)$$

First note that the photino propagator simplifies to

$$\frac{1 - \gamma_5}{2} \frac{i}{\not{k} - \mu} \frac{1 - \gamma_5}{2} = \frac{1 - \gamma_5}{2} \frac{i(\not{k} + \mu)}{k^2 - \mu^2} \frac{1 - \gamma_5}{2} = \frac{i\mu}{k^2 - \mu^2} \frac{1 - \gamma_5}{2} \quad (50)$$

because $\{\not{k}, \gamma_5\} = 0$. We also assume $M_L = M_R$, and then

$$\begin{aligned} i\mathcal{M}_{LR} = & 2e^3 A^* m \mu \bar{u}(p') \frac{1 - \gamma_5}{2} u(p) \int \frac{d^4k}{(2\pi)^4} \\ & \left[\frac{(p + p' - 2k)^\mu}{[(p' - k)^2 - M^2]^2 [(p - k)^2 - M^2]} + \frac{(p + p' - 2k)^\mu}{[(p' - k)^2 - M^2] [(p - k)^2 - M^2]^2} \right] \frac{1}{k^2 - \mu^2}. \end{aligned} \quad (51)$$

We use Feynman parameter integral to simplify the integrand as usual. By using the identity

$$\frac{1}{abc} = 2 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{1}{[z_1 a + z_2 b + z_3 c]^3}, \quad (52)$$

we act $-\partial/\partial b$ on both sides and find

$$\frac{1}{ab^2c} = 6 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{z_2}{[z_1 a + z_2 b + z_3 c]^4}. \quad (53)$$

Since the integrand we have has the form $\frac{1}{ab^2c} + \frac{1}{abc^2}$, we use

$$\frac{1}{ab^2c} + \frac{1}{abc^2} = 6 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{z_2 + z_3}{[z_1 a + z_2 b + z_3 c]^4}. \quad (54)$$

$$\begin{aligned} & \int \frac{d^4 k}{(2\pi)^4} \left[\frac{(p + p' - 2k)^\mu}{[(p' - k)^2 - M^2]^2 [(p - k)^2 - M^2]} + \frac{(p + p' - 2k)^\mu}{[(p' - k)^2 - M^2] [(p - k)^2 - M^2]^2} \right] \frac{1}{k^2 - \mu^2} \\ &= \int \frac{d^4 k}{(2\pi)^4} 6 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{(z_2 + z_3)(p + p' - 2k)^\mu}{[k^2 - 2z_2 p' \cdot k - 2z_3 p \cdot k - M^2]^4} \end{aligned} \quad (55)$$

where we ignored $(p')^2 = p^2 = m^2 \ll M^2$ in the denominator and set $\mu^2 = M^2$. We shift $k \rightarrow k + 2z_2 p' + 2z_3 p$, and we ignore $2p' \cdot p = 2m^2 - q^2 \ll M^2$ in the denominator

$$= \int \frac{d^4 k}{(2\pi)^4} 6 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{(1 - z_1)((1 - 2z_3)p + (1 - 2z_2)p' - 2k)^\mu}{[k^2 - M^2]^4}. \quad (56)$$

$d^4 k$ integration can be done using the general formula, and we find

$$= \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3) \frac{i}{(4\pi)^2} \frac{(1 - z_1)((1 - 2z_3)p + (1 - 2z_2)p')^\mu}{(M^2)^2}. \quad (57)$$

Now the integration volume is symmetric under $z_2 \leftrightarrow z_3$, and we can replace $(1 - 2z_3)$ by $[(1 - 2z_3) + (1 - 2z_2)]/2 = 1 - z_2 - z_3 = z_1$. Therefore,

$$= \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{i}{(4\pi)^2} \frac{(1 - z_1)z_1(p + p')^\mu}{(M^2)^2} = \frac{i}{(4\pi)^2} \frac{1}{12M^4} (p + p')^\mu. \quad (58)$$

By combining with the prefactors (and setting $\mu = M$), we find

$$i\mathcal{M}_{LR} = 2e^3 A^* m M \frac{i}{(4\pi)^2} \frac{1}{12M^4} \bar{u}(p') \frac{1 - \gamma_5}{2} u(p) (p + p')^\mu. \quad (59)$$

Now we use Gordon decomposition technique. As done in the class, we know that

$$\begin{aligned} m\gamma^\mu u(p) &= p^\mu u(p) - i\sigma^{\mu\nu} p_\nu u(p), \\ \bar{u}(p)m\gamma^\mu &= \bar{u}(p)p^\mu + \bar{u}(p)i\sigma^{\mu\nu} p_\nu. \end{aligned}$$

Therefore,

$$\begin{aligned} &\bar{u}(p') \frac{1-\gamma_5}{2} u(p) (p+p')^\mu \\ &= \bar{u}(p') \frac{1-\gamma_5}{2} (m\gamma^\mu + i\sigma^{\mu\nu} p_\nu) u(p) + \bar{u}(p') (m\gamma^\mu - i\sigma^{\mu\nu} p'_\nu) \frac{1-\gamma_5}{2} u(p) \\ &= \gamma^\mu \text{ piece} + \bar{u}(p') \frac{1-\gamma_5}{2} (-i\sigma^{\mu\nu} q_\nu) u(p). \end{aligned} \quad (60)$$

We used $q = p' - p$. The last piece is of our interest. We now find

$$i\mathcal{M}_{LR} = 2e^3 A^* m M \frac{i}{(4\pi)^2} \frac{1}{12M^4} \bar{u}(p') \frac{1-\gamma_5}{2} (-i\sigma^{\mu\nu} q_\nu) u(p). \quad (61)$$

Another amplitude of creating ϕ_R first and absorbing ϕ_L is obtained by simple replacements, $\frac{1-\gamma_5}{2} \rightarrow \frac{1+\gamma_5}{2}$, $A^* m \rightarrow Am$. Therefore, the total is given by

$$\begin{aligned} i\mathcal{M}_{\text{total}} &= 2e^3 m \frac{i}{(4\pi)^2} \frac{1}{12M^4} \bar{u}(p') (-i\sigma^{\mu\nu} q_\nu) (\text{Re}A + i\gamma_5 \text{Im}A) u(p) \\ &= i \frac{\alpha}{4\pi} \frac{m}{M^3} \frac{e}{6} \bar{u}(p') (-i\sigma^{\mu\nu} q_\nu) (\text{Re}A + i\gamma_5 \text{Im}A) u(p). \end{aligned} \quad (62)$$

Now comes the interpretation of this amplitude as the electric dipole moment of the electron. First notice that the amplitudes can be regarded as the matrix element of an effective Feynman rule $-iH_{\text{eff}}$. We focus on the $\sigma^{\mu\nu}\gamma_5$ piece. The effective Hamiltonian can be read off as

$$\langle p' | H_{\text{eff}} | p \rangle = \frac{\alpha}{4\pi} \frac{m}{M^3} \frac{e}{6} \bar{u}(p') (i\sigma^{\mu\nu} q_\nu) (i\gamma_5 \text{Im}A) u(p) A_\mu(q). \quad (63)$$

Here, $A_\mu(q)$ is the vector potential in the momentum space. Since we are applying an electric field, let us take $A^0 \neq 0$ and other components zero. Then the matrix we need to understand is $\sigma^{0i}\gamma_5$. Taking Pauli–Dirac representation suitable for non-relativistic limit, we find

$$\sigma^{0i}\gamma_5 = i\gamma^0\gamma^i\gamma_5 = i\alpha^i\gamma_5 = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}. \quad (64)$$

Non-relativistic electron states occupy only the upper two components, and we keep only $i\sigma^i$ in our expression. Then

$$\langle p'|H_{eff}|p\rangle = \frac{\alpha}{4\pi} \frac{m}{M^3} \frac{e}{6} \text{Im}A \bar{u}(p') \sigma^i u(p) i q_i A_0(q). \quad (65)$$

Since $i q_i A_0 = -i q^i A_0 = -\nabla^i A_0 = E^i$, the matrix element is then

$$\langle p'|H_{eff}|p\rangle = \bar{u}(p') \left[\frac{\alpha}{4\pi} \frac{m}{M^3} \frac{e}{6} \text{Im}A \vec{\sigma} \cdot \vec{E} \right] u(p). \quad (66)$$

Clearly this is an electric dipole moment

$$d_e = \frac{\alpha}{4\pi} \frac{m}{M^3} \frac{e}{6} \text{Im}A. \quad (67)$$

Taking $A = M e^{i\phi}$, we can write $\text{Im}A = M \sin \phi$.

$$d_e = \frac{\alpha}{4\pi} \frac{m}{M^2} \frac{e}{6} \sin \phi = 9.76 \times 10^{-28} e \text{ cm} \times \sin \phi \left(\frac{\text{TeV}}{M} \right)^2. \quad (68)$$

From Gene's result $d_e = (0.18 \pm 0.12 \pm 0.10) \times 10^{-26} e \text{ cm}$, we combine statistical and systematic errors in quadrature $\sqrt{(0.12)^2 + (0.10)^2} = 0.16$. Taking 95% confidence level (*i.e.*, two sigma), we find $|d_e| < 0.50 \times 10^{-26} e \text{ cm}$. Comparing this to the expression found from supersymmetry above, we conclude

$$M > 440 \text{ GeV} (\sin \phi)^{1/2}. \quad (69)$$

If the phase of A is order unity $\sin \phi \sim 1$. This is actually one of the most stringent constraints on supersymmetry.