HW #1, due Sep 3

1. Show that the Lagrangian of the Schrödinger field ψ

$$L = \int d^3x\, \mathcal{L}, \qquad \mathcal{L} = \psi^\dagger \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \vec{\nabla}^2}{2m} \right) \psi - \frac{\lambda}{2} :(\psi^\dagger \psi)^2: \tag{1}$$

is invariant under the following transformations:

(a) phase rotation $\psi'(\vec{x}) = e^{i\theta}\psi(\vec{x})$, with a constant θ.

(b) spatial translation $\psi'(\vec{x}) = \psi(\vec{x} + \vec{a})$ with a constant vector $\vec{a}$.

(c) Gallilean transformation

$$\psi'(\vec{x}, t) = e^{-im\vec{v}^2 t/2\hbar} e^{im\vec{v}\cdot\vec{x}/\hbar} \psi(\vec{x} - \vec{v}t, t) \tag{2}$$

with a constant vector $\vec{v}$.

Rem The invariance of the Lagrangian means

$$L' = \int d^3x\, \mathcal{L}(\psi') = \int d^3x\, \mathcal{L}(\psi) = L \tag{3}$$

up to surface terms and total time derivatives.

2. The three-particle state

$$|\Psi\rangle = \int d^3x\, d^3y\, d^3z\, \Psi(\vec{x}, \vec{y}, \vec{z}, t) \psi^\dagger(\vec{x}) \psi^\dagger(\vec{y}) \psi^\dagger(\vec{z}) |0\rangle \tag{4}$$

satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle, \tag{5}$$

where the Hamiltonian H is derived from the Lagrangian Eq. (1). Show that the coefficient function $\Psi(\vec{x}, \vec{y}, \vec{z}, t)$ satisfies the equation

$$\begin{aligned} & i\hbar \frac{\partial}{\partial t} \Psi(\vec{x}, \vec{y}, \vec{z}, t) \\ &= \left(-\frac{\hbar^2 \vec{\nabla}_x^2}{2m} - \frac{\hbar^2 \vec{\nabla}_y^2}{2m} - \frac{\hbar^2 \vec{\nabla}_z^2}{2m} + \lambda \delta^3(x-y) + \lambda \delta^3(y-z) + \lambda \delta^3(z-x) \right) \Psi(\vec{x}, \vec{y}, \vec{z}, t). \end{aligned}$$