Physics 229a

Solution Set # 1

(1) Consider the Lagrangian

$$L = \int d^3x\, \mathcal{L}(\psi(x))$$

$$\mathcal{L}(\psi(x)) = \psi^\dagger(x)\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2m}\right)\psi(x) - \frac{\lambda}{2}:(\psi^\dagger(x)\psi(x))^2:$$

Now I'll verify that L has the following symmetries

(a) Phase rotation $\psi'(\vec{x}) = e^{i\theta}\psi(\vec{x})$

$$\psi^{\dagger\prime}(\vec{x}) = (\psi'(\vec{x}))^\dagger = (e^{i\theta}\psi(\vec{x}))^\dagger = e^{-i\theta}\psi^\dagger(\vec{x})$$

Plugging these transformations into the Lagrangian density $\mathcal{L}$:

$$\mathcal{L}(\psi'(\vec{x})) = \psi^{\dagger\prime}(x)\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2m}\right)\psi'(x) - \frac{\lambda}{2}:(\psi^{\dagger\prime}(x)\psi'(x))^2:$$

$$= \cancel{e^{-i\theta}}\psi'(x)\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2m}\right)\cancel{e^{i\theta}}\psi(x)$$

$$- \frac{\lambda}{2}:(\cancel{e^{-i\theta}}\psi^\dagger(x)\cancel{e^{i\theta}}\psi(x))^2:$$

$$= \mathcal{L}(\psi(\vec{x}))$$

Therefore $L' = \int d^3x\, \mathcal{L}(\psi'(\vec{x})) = \int d^3x\, \mathcal{L}(\psi(\vec{x})) = L$

(b) $\psi'(\vec{x}) = \psi(\vec{x}+\vec{a})$ with constant $\vec{a}$

$$L' = \int d^3\vec{x}\ \mathcal{L}(\psi'(\vec{x}))$$
$$= \int d^3\vec{x}\ \mathcal{L}(\psi(\vec{x}+\vec{a}))$$
$$= \int d^3(\vec{x}+\vec{a})\ \mathcal{L}(\psi(\vec{x}+\vec{a}))$$

(I've used translation invariance of the measure $d^3\vec{x}$)

$$= \int d^3\vec{x}'\ \mathcal{L}(\psi(\vec{x}'))$$

(where $\vec{x}' = \vec{x}+\vec{a}$)

$$= L$$

(c) $\psi'(\vec{x},t) = e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\, \psi(\vec{x}-\vec{v}t, t)$

$$\mathcal{L}(\psi'(\vec{x},t)) = e^{im\vec{v}^2 t/2\hbar}\, e^{-im\vec{v}\cdot\vec{x}/\hbar}\, \psi^\dagger(\vec{x}-\vec{v}t, t)$$
$$\cdot\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2m}\right) e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\, \psi(\vec{x}-vt, t)$$
$$-\frac{\lambda}{2}:\left(\cancel{e^{im\vec{v}^2 t/2\hbar}\, e^{-im\vec{v}\cdot\vec{x}/\hbar}}\, \psi^\dagger(\vec{x}-\vec{v}t,t)\, \cancel{e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}}\, \psi(\vec{x}-\vec{v}t,t)\right)^2$$

I'll evaluate the above derivatives seperately

$$\frac{\partial}{\partial t} e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\, \psi(\vec{x}-\vec{v}t, t)$$
$$= e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\left(-\frac{im\vec{v}^2}{2\hbar}\psi(\vec{x}-vt,t) - \vec{v}\cdot\vec{\nabla}_y\psi(\vec{y},t)\Big|_{y=\vec{x}-\vec{v}t} + \frac{\partial}{\partial t'}\psi(\vec{x}-\vec{v}t,t')\Big|_{t'=t}\right)$$

and the other derivative term is

$$\vec{\nabla}^2 e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\, \psi(\vec{x}-\vec{v}t,t)$$
$$= e^{-im\vec{v}^2 t/2\hbar}\, e^{im\vec{v}\cdot\vec{x}/\hbar}\left(\left(\frac{im}{\hbar}\right)^2\vec{v}^2 + \frac{2im\vec{v}\cdot\vec{\nabla}}{\hbar} + \vec{\nabla}^2\right)\psi(\vec{x}-\vec{v}t,t)$$

Note $\vec{\nabla}_y\psi(\vec{y},t)\big|_{\vec{y}=\vec{x}-\vec{v}t} = \vec{\nabla}_x\psi(\vec{x}-\vec{v}t,t)$

Combining these results:

$$\mathcal{L}(\psi'(\vec{x},t)) = e^{\cancel{im\vec{v}^2 t/2\hbar}}\, e^{\cancel{-im\vec{v}\cdot\vec{x}/\hbar}}\, \psi^{\dagger}(\vec{x}-\vec{v}t,t)$$

$$e^{\cancel{-im\vec{v}^2 t/2\hbar}}\, e^{\cancel{im\vec{v}\cdot\vec{x}/\hbar}} \Big(\cancel{\frac{m\vec{v}^2}{2}} - \cancel{i\hbar\vec{v}\cdot\vec{\nabla}} + i\hbar\frac{\partial}{\partial t}$$

$$-\cancel{\frac{m\vec{v}^2}{2}} + \cancel{i\hbar\vec{v}\cdot\vec{\nabla}} + \frac{\hbar^2\vec{\nabla}^2}{2m} \Big)\psi(\vec{x}-\vec{v}t,t)$$

$$-\frac{\lambda}{2} :(\psi^{\dagger}(\vec{x}-\vec{v}t,t)\psi(\vec{x}-\vec{v}t,t))^2:$$

$$= \mathcal{L}(\psi(\vec{x}-\vec{v}t,t))$$

Using part (b) with $\vec{a} = -\vec{v}t$ this is identified as a spatial translation and so is a symmetry of the Lagrangian

$$L' = L.$$

(2) $|\Psi\rangle = \int d^3x\, d^3y\, d^3z\; \Psi(\vec{x},\vec{y},\vec{z},t)\, \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$

$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$ (Schrodinger's Equation)

First I'll find H

$\Pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})} = i\hbar\, \psi^\dagger(\vec{x})$

$\mathcal{H}(\psi,\Pi) = \Pi(\vec{x})\,\dot{\psi}(\vec{x}) - \mathcal{L}(\psi(\vec{x}))$

$= -\psi^\dagger(\vec{x})\, \frac{\hbar^2\vec{\nabla}^2}{2m}\, \psi(\vec{x}) + \frac{\lambda}{2}\, \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{x})\,\psi(\vec{x})\,\psi(\vec{x})$

$H = \int d^3x\; \mathcal{H}(\psi(\vec{x}), \Pi(\vec{x}))$

Plugging H into Schrodinger's Equation

$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \int d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\, \left(i\hbar \frac{\partial}{\partial t}\Psi(\vec{x},\vec{y},\vec{z},t)\right) \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$

$= H|\Psi\rangle$

$= \int d^3\vec{w}\, d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\, \left(-\psi^\dagger(w)\, \frac{\hbar^2\vec{\nabla}_w^2}{2m}\, \psi(\vec{w}) + \frac{\lambda}{2}\, \psi^\dagger(\vec{w})\,\psi^\dagger(\vec{w})\,\psi(\vec{w})\,\psi(\vec{w})\right)$

(*) $\cdot\, \Psi(\vec{x},\vec{y},\vec{z},t)\, \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})\,|0\rangle \equiv \underset{\substack{\uparrow\\ 1^{st}\text{ term}\\ (K.E.)}}{①} + \underset{\substack{\nwarrow\\ 2^{nd}\text{ term}\\ (\text{interaction})}}{②}$

I'll evaluate this a piece at a time. First:

$\psi(\vec{w})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})\,|0\rangle = [\psi(\vec{w}), \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})]|0\rangle$

$= [\psi(\vec{w}), \psi^\dagger(\vec{x})]\,\psi^\dagger(\vec{y})\,\psi^\dagger(z)|0\rangle$

$+ \psi^\dagger(\vec{x})\,[\psi(\vec{w}), \psi^\dagger(\vec{y})]\,\psi^\dagger(z)|0\rangle$

$+ \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,[\psi(\vec{w}), \psi^\dagger(z)]|0\rangle$

$= \left(\delta^3(\vec{w}-\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z}) + \delta^3(\vec{w}-\vec{y})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{z})\right.$

$\left. + \delta^3(\vec{w}-\vec{z})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\right)|0\rangle$

Second:

$$\psi(\vec{w})\,\psi(\vec{w})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$

$$= \psi(\vec{w})\,[\psi(\vec{w}),\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})]\,|0\rangle$$

$$= [\psi(\vec{w}),\ \delta^3(\vec{w}-\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z}) + \delta^3(\vec{w}-\vec{y})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{z}) + \delta^3(\vec{w}-\vec{z})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\ ]|0\rangle$$

$$= \delta^3(\vec{w}-\vec{x})\Big([\psi(\vec{w}),\psi^\dagger(\vec{y})]\,\psi^\dagger(\vec{z}) + \psi^\dagger(\vec{y})\,[\psi(\vec{w}),\psi^\dagger(\vec{z})]\Big)|0\rangle$$
$$+\,\delta^3(\vec{w}-\vec{y})\Big([\psi(\vec{w}),\psi^\dagger(\vec{x})]\,\psi^\dagger(\vec{z}) + \psi^\dagger(\vec{x})\,[\psi(\vec{w}),\psi^\dagger(\vec{z})]\Big)|0\rangle$$
$$+\,\delta^3(\vec{w}-\vec{z})\Big([\psi(\vec{w}),\psi^\dagger(\vec{x})]\,\psi^\dagger(\vec{y}) + \psi^\dagger(\vec{x})\,[\psi(\vec{w}),\psi^\dagger(\vec{y})]\Big)|0\rangle$$

$$= 2\,\delta^3(\vec{w}-\vec{x})\,\delta^3(\vec{w}-\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$
$$+\,2\,\delta^3(\vec{w}-\vec{x})\,\delta^3(\vec{w}-\vec{z})\,\psi^\dagger(\vec{y})|0\rangle$$
$$+\,2\,\delta^3(\vec{w}-\vec{y})\,\delta^3(\vec{w}-\vec{z})\,\psi^\dagger(\vec{x})|0\rangle$$

And so the first term of $(*)$ is

$$① = \int d^3\vec{w}\,d^3\vec{x}\,d^3\vec{y}\,d^3\vec{z}\,\Big(-\psi^\dagger(\vec{w})\frac{\hbar^2\vec{\nabla}_w^2}{2m}\psi(\vec{w})\Big)\Psi(\vec{x},\vec{y},\vec{z},t)\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$

$$= -\int d^3\vec{w}\,d^3\vec{x}\,d^3\vec{y}\,d^3\vec{z}\ \psi^\dagger(\vec{w})\frac{\hbar^2\vec{\nabla}_w^2}{2m}\delta^3(\vec{w}-\vec{x})\ \Psi(\vec{x},\vec{y},\vec{z},t)\ \psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$

+ cyclic perms of (xyz)

$$\overset{IBP}{=} -\frac{\hbar^2}{2m}\int d^3\vec{w}\,d^3\vec{x}\,d^3\vec{y}\,d^3\vec{z}\,\Big(\vec{\nabla}_w^2\,\psi^\dagger(\vec{w})\Big)\delta^3(\vec{w}-\vec{x})\ \Psi(\vec{x},\vec{y},\vec{z},t)\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$

+ permutations

$$= -\frac{\hbar^2}{2m}\int d^3\vec{x}\,d^3\vec{y}\,d^3\vec{z}\,\Big(\vec{\nabla}_x^2\,\psi^\dagger(\vec{x})\Big)\Psi(\vec{x},\vec{y},\vec{z},t)\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle + \text{perms}$$

$$\overset{IBP}{=} -\frac{\hbar^2}{2m}\int d^3\vec{x}\,d^3\vec{y}\,d^3\vec{z}\,\Big(\vec{\nabla}_x^2\,\Psi(\vec{x},\vec{y},\vec{z},t)\Big)\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle + \text{perms}$$

The second term of ($*$) is

$$② = \frac{\lambda}{2}\int d^3\vec{w}\, d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\ \psi^\dagger(\vec{w})\,\psi^\dagger(\vec{w})\,\psi(\vec{w})\,\psi(\vec{w})\,\Psi(\vec{x},\vec{y},\vec{z},t)\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$$

$$= \frac{\lambda}{\cancel{2}}\cdot\cancel{2}\int d^3\vec{w}\, d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\ \psi^\dagger(\vec{w})\,\psi^\dagger(\vec{w})\,\delta^3(\vec{w}-\vec{x})\,\delta^3(\vec{w}-\vec{y})\,\psi^\dagger(\vec{z})\,\Psi(\vec{x},\vec{y},\vec{z},t)|0\rangle + \text{perms}$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\ \Psi(\vec{x},\vec{y},\vec{z},t)\left(\delta^3(\vec{x}-\vec{y})+\delta^3(\vec{x}-\vec{z})+\delta^3(\vec{y}-\vec{z})\right)$$
$$\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})\,|0\rangle$$

Combining everything

$$\int d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\left(i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},\vec{y},\vec{z},t)\right)\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})\,|0\rangle$$
$$= \int d^3\vec{x}\, d^3\vec{y}\, d^3\vec{z}\left(-\frac{\hbar^2\vec{\nabla}_x^2}{2m}-\frac{\hbar^2\vec{\nabla}_y^2}{2m}-\frac{\hbar^2\vec{\nabla}_z^2}{2m}+\lambda\delta^3(\vec{x}-\vec{y})+\lambda\delta^3(\vec{x}-\vec{z})+\lambda\delta^3(\vec{y}-\vec{z})\right)$$
$$\cdot\,\Psi(\vec{x},\vec{y},\vec{z},t)\ \psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})\,|0\rangle.$$

The states $\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi^\dagger(\vec{z})|0\rangle$ are linearly independent so I can match their coefficients:

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},\vec{y},\vec{z},t) = \left(-\frac{\hbar^2\vec{\nabla}_x^2}{2m}-\frac{\hbar^2\vec{\nabla}_y^2}{2m}-\frac{\hbar^2\vec{\nabla}_z^2}{2m}+\lambda\delta^3(\vec{x}-\vec{y})+\lambda\delta^3(\vec{x}-\vec{z})+\lambda\delta^3(\vec{y}-\vec{z})\right)\Psi(\vec{x},\vec{y},\vec{z},t)}$$