Problem Set # 6

1. <u>Toy Dirac Model</u> $\quad L = \bar{\Psi}(i\sigma_3 \partial_t - m)\psi$

Here ψ is a two-component dynamical variable, and $\bar{\psi} = \psi^\dagger \sigma_3$.

The conjugate momentum is $\quad \pi = \frac{\partial L}{\partial \dot{\psi}} = \bar{\psi}\, i\sigma_3 = i\psi^\dagger$.

Quantize with anti-commutation relations: $\{\psi_\alpha, i\psi_\beta^\dagger\} = i\delta_{\alpha\beta}$, etc.

The equation of motion is $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{\psi}}}\right)^0 - \frac{\partial L}{\partial \bar{\psi}} = 0 \quad \Rightarrow \quad (i\sigma_3 \partial_t - m)\psi = 0$

(1) Positive & negative energy solutions.

The equation of motion implies that each component of ψ obeys a harmonic oscillator equation (with frequency $\pm\frac{mc^2}{\hbar}$)

$$(-i\sigma_3 \partial_t - m)(i\sigma_3 \partial_t - m)\psi = 0$$

$$(\partial_t^2 + m^2)\psi = 0$$

The solutions $\psi = e^{-iEt}$ can have positive or negative frequencies, and the "energy-momentum" relation has been reduced to

$$-E^2 + m^2 = 0$$
$$E = \pm m$$

<u>Positive energy</u> $\quad$ Try $\quad \psi(t) = u\, e^{-imt}$

$$(i\sigma_3 \partial_t - m)\psi = (m\sigma_3 - m)u e^{-imt} = \begin{pmatrix} 0 & 0 \\ 0 & -2m \end{pmatrix} u e^{-imt} = 0.$$

So $\underline{u = \binom{1}{0}}$, with normalization chosen s.t. $\underline{u^\dagger u = 1}$.

<u>Negative energy</u> $\quad$ Try $\quad \psi(t) = v\, e^{imt}$

$$(i\sigma_3 \partial_t - m)\psi = (-m\sigma_3 - m)v\, e^{imt} = \begin{pmatrix} -2m & 0 \\ 0 & 0 \end{pmatrix} v e^{imt} = 0$$

$\longrightarrow \quad \underline{v = \binom{0}{1}}, \qquad \underline{v^\dagger v = 1}.$

(2) Anti-commutation relations amongst the mode operators.

Expand ψ as

$$\psi_{(t)} = a u e^{-imt} + b^\dagger v e^{imt}$$
$$\psi^\dagger_{(t)} = a^\dagger u^\dagger e^{imt} + b v^\dagger e^{-imt}$$

So the mode ops. are

$$a = u^\dagger \psi e^{imt} = \psi_1 e^{imt}$$
$$b^\dagger = v^\dagger \psi e^{-imt} = \psi_2 e^{-imt}$$

$$a^\dagger = \psi^\dagger u e^{-imt} = \psi_1^\dagger e^{-imt}$$
$$b = \psi^\dagger v e^{imt} = \psi_2^\dagger e^{imt}$$

Then using the anti-commutation relations $\{\psi_\alpha, \psi^\dagger_\beta\} = \delta_{\alpha\beta}$,

$$\{a, a^\dagger\} = \{\psi_1 e^{imt}, \psi_1^\dagger e^{-imt}\} = \{\psi_1, \psi_1^\dagger\} = \delta_{11} = 1$$
$$\{b, b^\dagger\} = \{\psi_2^\dagger, \psi_2\} = \delta_{22} = 1$$

Also, $\{a, b\} = \{a, b^\dagger\} = \{a^\dagger, b\} = \{a^\dagger, b^\dagger\} = 0$ all follow
$\{a, a\} = \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0$

from $\{\psi_\alpha, \psi_\beta\} = \{\psi^\dagger_\alpha, \psi^\dagger_\beta\} = 0.$

Since $\{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0$, $(a^\dagger)^2 = (b^\dagger)^2 = 0$. Thus the Hilbert space has only the four states

$$|0\rangle, \quad a^\dagger|0\rangle, \quad b^\dagger|0\rangle, \quad a^\dagger b^\dagger|0\rangle$$

as a basis.

(3) The Hamiltonian is $H_0 = \pi\dot{\psi} - L$

We said above that $\pi = \frac{\partial L}{\partial\dot{\psi}} = \bar{\psi}\, i\sigma_3 \; (= i\psi^\dagger)$.

$$H_0 = (\bar{\psi}\, i\sigma_3)\partial_t\psi - \left(\cancel{\bar{\psi}\, i\sigma_3\partial_t\psi} - m\bar{\psi}\psi\right)$$

$$= m\bar{\psi}\psi$$

$$= m\psi^\dagger\sigma_3\psi$$

$$= m\left(a^\dagger u^\dagger e^{+imt} + b v^\dagger e^{-imt}\right)\sigma_3\left(a u e^{-imt} + b^\dagger v e^{imt}\right)$$

use $u^\dagger\sigma_3 u = 1$, $v^\dagger\sigma_3 v = -1$,
$u^\dagger\sigma_3 v = v^\dagger\sigma_3 u = 0$.

$$H_0 = m(a^\dagger a - b b^\dagger)$$

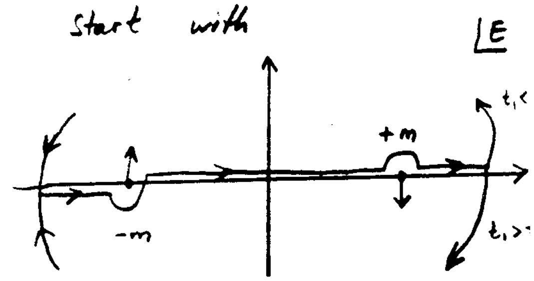
$\{b, b^\dagger\} = 1$

$$H_0 = m(a^\dagger a + b^\dagger b) - m$$

(4) Feynman Propagator $\quad S_F^{\alpha\beta}(t_1 - t_2) = \langle 0|\, T\,\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)\,|0\rangle$

It's easiest to work in both directions. Start with

$$\int_{-\infty}^{\infty}\frac{dE}{2\pi}\,\frac{i(E\sigma_3 + m)_{\alpha\beta}}{E^2 - m^2 + i\epsilon}\, e^{-iE(t_1 - t_2)} .$$



The $+i\epsilon$ prescription tells us to take the contour as shown. If $t_1 - t_2 > 0$ we can close the contour in the LHP, picking up the $+m$ pole with an extra $-$ sign (since the contour is clockwise). If $t_1 - t_2 < 0$, close the contour in the UHP, picking up the $-m$ pole with a counterclockwise (positive) contour.

$$\int = 2\pi i\left(\overset{L}{-}\theta(t_1 - t_2)\,\frac{i}{2\pi}\,\frac{(+m\sigma_3 + m)_{\alpha\beta}}{(+m+m)}\, e^{-i(+m)(t_1 - t_2)} \overset{U}{+} \theta(t_2 - t_1)\,\frac{i}{2\pi}\,\frac{(-m\sigma_3 + m)_{\alpha\beta}}{(-m-m)}\, e^{-i(-m)(t_1 -} \right.$$

$$= \theta(t_1 - t_2)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_{\alpha\beta} e^{-im(t_1 - t_2)} + \theta(t_2 - t_1)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_{\alpha\beta} e^{+im(t_1 - t_2)}$$

(4) cont. We can write the result of the calculation simply as

$$\int = \begin{pmatrix} \theta(t_1-t_2) & 0 \\ 0 & \theta(t_2-t_1) \end{pmatrix}_{\alpha\beta} e^{-im|t_1-t_2|}$$

Next we work on

$$S_F^{\alpha\beta}(t_1-t_2) = \langle 0 |\, T\, \psi_\alpha(t_1)\, \bar{\psi}_\beta(t_2)\, | 0 \rangle$$

$$= \theta(t_1-t_2)\, \langle 0 |\, (a u_\alpha e^{-imt_1} + \cancel{b^\dagger v_\alpha} e^{imt_1})(a^\dagger (u^\dagger\sigma_3)_\beta e^{+imt_2} + \cancel{b (v^\dagger\sigma_3)_\beta} e^{-imt_2})\, | 0 \rangle$$

Time-ordered product for anti-commuting operators.

$$- \theta(t_2-t_1)\, \langle 0 |\, (\cancel{a^\dagger (u^\dagger\sigma_3)_\beta} e^{+imt_2} + b (v^\dagger\sigma_3)_\beta e^{-imt_2})(\cancel{a u_\alpha e^{-imt_1}} + b^\dagger v_\alpha e^{imt_1})\, | 0 \rangle$$

$$= \theta(t_1-t_2)\, u_\alpha u_\beta \langle 0 | a a^\dagger | 0 \rangle e^{-im(t_1-t_2)} - \theta(t_2-t_1)\, (-v_\beta v_\alpha) \langle 0 | b b^\dagger | 0 \rangle e^{im(t_1-t_2)}$$

$$= \theta(t_1-t_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} e^{-im(t_1-t_2)} + \theta(t_2-t_1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta} e^{im(t_1-t_2)}$$

$$S_F^{\alpha\beta}(t_1-t_2) = \begin{pmatrix} \theta(t_1-t_2) & 0 \\ 0 & \theta(t_2-t_1) \end{pmatrix}_{\alpha\beta} e^{-im|t_1-t_2|}$$

This is the same as the expression above for the value of the integral representation.

$$\therefore \quad S_F^{\alpha\beta}(t_1-t_2) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{i\, e^{-iE(t_1-t_2)}}{E\sigma_3 - m + i\epsilon},$$

where we used $(E\sigma_3+m)(E\sigma_3-m) = E^2-m^2$ to rewrite the integrand.

(5). Calculate $\langle 0 | T\, \psi(t_1)\, \bar{\psi}(t_2)\, \psi(t_3)\, \bar{\psi}(t_4) | 0 \rangle$ when $t_1 > t_2 > t_3 > t_4$.

<u>(1) Wick's Theorem</u>

There are only two possible ways to take contractions:

$$\langle 0 | T\, \psi_\alpha(t_1)\, \bar{\psi}_\beta(t_2)\, \psi_\gamma(t_3)\, \bar{\psi}_\delta(t_4) | 0 \rangle =$$

$$\langle 0 | \underbrace{\underbrace{\psi \bar{\psi}}\ \underbrace{\psi \bar{\psi}}}_{(-1)^0} | 0 \rangle + \langle 0 | \underbrace{\psi\, \underbrace{\bar{\psi}\, \psi}\, \bar{\psi}}_{(-1)^3} | 0 \rangle$$

$$= S_F^{\alpha\beta}(t_1 - t_2)\, S_F^{\gamma\delta}(t_3 - t_4) - S_F^{\alpha\delta}(t_1 - t_4)\, S_F^{\gamma\beta}(t_3 - t_2)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\gamma\delta} e^{-im(t_1 - t_2)}\, e^{-im(t_3 - t_4)}$$

$$- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\delta} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\gamma\beta} e^{-im(t_1 - t_4)}\, e^{\overset{+}{-}im(t_2 - t_3)}$$

$t_2 > t_3 :\ \theta(t_2 - t_3) = 1$

<u>(2) Creation / annihilation ops.</u>

$$= \langle 0 | \left(a u_\alpha e^{-imt_1} + \cancel{b^\dagger v_\alpha e^{imt_1}}\right)\left(a^\dagger \bar{u}_\beta e^{imt_2} + b \bar{v}_\beta e^{-imt_2}\right)\left(a u_\gamma e^{-imt_3} + b^\dagger v_\gamma e^{imt_3}\right) \cdot$$
$$\left(a^\dagger \bar{u}_\delta e^{imt_4} + \cancel{b \bar{v}_\delta e^{-imt_4}}\right) | 0 \rangle$$

Two terms contribute.
$\bar{u} = u^\dagger$, $\bar{v} = -v^\dagger$

$$= u_\alpha u_\beta u_\gamma u_\delta \langle 0 | a a^\dagger a a^\dagger | 0 \rangle\, e^{+im(-t_1 + t_2 - t_3 + t_4)}$$
$$+ u_\alpha \bar{v}_\beta v_\gamma u_\delta \langle 0 | a b b^\dagger a^\dagger | 0 \rangle\, e^{im(-t_1 - t_2 + t_3 + t_4)}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\gamma\delta} e^{-im(t_1 - t_2)}\, e^{-im(t_3 - t_4)}$$

$$- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\delta} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\gamma\beta} e^{-im(t_1 - t_4)}\, e^{-i(t_2 - t_3)}$$

This is the same as obtained by wick's theorem above. We sum over

(5) cont. Sum over $\beta=\gamma$ to get

$$\langle 0|T\,\psi(t_1)\bar\psi(t_2)\psi(t_3)\bar\psi(t_4)|0\rangle = \begin{pmatrix}1&0\\0&0\end{pmatrix} e^{-im(t_1-t_2)}e^{-im(t_3-t_4)} - \begin{pmatrix}1&0\\0&0\end{pmatrix} e^{-im(t_1-t_4)}e^{-im(t_2-t_3)}$$

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(6) Add a time-dependent perturbation. Calculate $\langle 0(+\infty)|0(-\infty)\rangle$.

$$H = H_0 + V(t)\,, \qquad V = f(t)\,\psi^\dagger\sigma_1\psi\,.$$

$V^\dagger = V \Rightarrow f^*(t) = f(t) \Rightarrow f(t)$ real. We also assume that $f(t)\to 0$ at $\pm\infty$.

$$\langle 0(\infty)|0(-\infty)\rangle = {}_I\langle 0|T\,e^{-i\int_{-\infty}^{\infty}V_I(t)\,dt}|0\rangle_I$$

$$= {}_I\langle 0|T\left[1 - i\int_{-\infty}^{\infty}V_I(t)\,dt - \tfrac{1}{2}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt''\,V_I(t')V_I(t'') + \cdots\right]|0\rangle_I$$

$$= {}_I\langle 0|0\rangle_I - i\,{}_I\langle 0|\int_{-\infty}^{\infty}V_I(t)\,dt\,|0\rangle_I - \tfrac{1}{2}\,{}_I\langle 0|T\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt''\,V_I(t')V_I(t'')|0\rangle_I \cdots$$

$o(f^0)$: $\quad {}_I\langle 0|0\rangle_I = 1$

Now
$$\begin{aligned} V &= f(t)\,\psi^\dagger\sigma_1\psi \\ &= f(t)\,\bar\psi\,\sigma_3\sigma_1\psi \\ &= f(t)\,\bar\psi\,i\sigma_2\psi \\ &= f(t)\,(\bar\psi_1\ \bar\psi_2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\end{aligned}$$

$$V = f(t)\left(\bar\psi_1\psi_2 - \bar\psi_2\psi_1\right) = f(t)\left(\psi_1\bar\psi_2 - \psi_2\bar\psi_1\right)$$

$$V_I = f(t)\left(\psi_{1I}\bar\psi_{2I} - \psi_{2I}\bar\psi_{1I}\right)$$

$o(f^1)$: $\quad {}_I\langle 0|V_I|0_I\rangle = f(t)\,{}_I\langle 0|T\left(\psi_{1I}(t)\bar\psi_{2I}(t) - \psi_{2I}(t)\bar\psi_{1I}(t)\right)|0\rangle_I$

$$= \overset{\to 0}{S_F^{12}(t-t)} - \overset{\to 0}{S_F^{21}(t-t)}$$

$$= 0$$

(6) (cont.) The overlap vanishes at order $o(f^1)$, so we go to $o(f^2)$:

$$o(f^2): \quad {}_I\langle 0|\, T\, V_I(t')\, V_I(t'')\, |0\rangle_I$$

$$= f(t')\, f(t'')\; {}_I\langle 0|\, T \left(\psi_{1I}(t')\, \bar\psi_{2I}(t') - \psi_{2I}(t')\, \bar\psi_{1I}(t') \right) \times \left(\psi_{1I}(t'')\, \bar\psi_{2I}(t'') - \psi_{2I}(t'')\, \bar\psi_{1I}(t'') \right) |0\rangle_I$$

We know from above that contractions which give S_F^{12} or S_F^{21} vanish, so the only contributions are

$$= f(t')\, f(t'') \Big(-\, {}_I\langle 0|\, T\, \psi_{1I}(t')\, \bar\psi_{2I}(t')\, \psi_{2I}(t'')\, \bar\psi_{1I}(t'')\, |0\rangle_I$$

$$-\, {}_I\langle 0|\, T\, \psi_{2I}(t')\, \bar\psi_{1I}(t')\, \psi_{1I}(t'')\, \bar\psi_{2I}(t'')\, |0\rangle_I \Big)$$

$$= +f(t')\, f(t'') \left(S_F^{11}(t'-t'')\, S_F^{22}(t''-t') \cdot 2 \right)$$

$${}_I\langle 0|\, T\, e^{-i\int_{-\infty}^{\infty} V_I(t)\,dt}\, |0\rangle_I = 1 - \frac{1}{2}\, 2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, f(t')\, f(t'')\, S_F^{11}(t'-t'')\, S_F^{22}(t''-t'$$

$$= 1 - \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, f(t')\, f(t'')\, e^{2im|t'-t''|}\, \theta(t'-t'')$$

(7) $_{out}\langle a|\, O(t_1)\, |0\rangle$ matrix element.

$$\int_{-\infty}^{\infty} dt\ e^{imt}\,(-i)\,\bar{u}\,(i\sigma_3\,\partial_t - m)\,\langle 0|\,T\,O(t_1)\,\psi(t)\,|0\rangle$$

$$= e^{imt}(-i)\,\bar{u}\,(i\sigma_3)\,\langle 0|\,T\,O(t_1)\,\psi(t)\,|0\rangle\Big|_{t=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty} e^{imt}(-i)\,\bar{u}\,(i\sigma_3(-1)im - m)\,\langle 0|\,T\,O(t_1)\,\psi(t)\,|0\rangle$$

$$= \langle 0|\,T\,O(t_1)\,\overbrace{(e^{imt}u^{\dagger}\,\psi(t))}^{\text{reduces to } a \text{ at } \pm\infty}\,|0\rangle\Big|_{t=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty} e^{imt}(-i)\ \cancelto{0}{(1\ 0)\begin{pmatrix}0&0\\0&-2m\end{pmatrix}}\ \langle 0|\,T\ \ |0\rangle$$

bosonic

$$= \langle 0|\ \overbrace{a(+\infty)\,O(t_1)}^{+\text{ sign}} - O(t_1)\,a(-\infty)\ |0\rangle$$

$$= {}_{out}\langle a|\ O(t_1)\ |0\rangle$$

(8) $_{out}\langle b|\, O(t_1)\, |0\rangle$ matrix element.

$$\int_{-\infty}^{\infty} dt\ \langle 0|\,T\,O(t_1)\,\bar{\psi}_\beta(t)\,|0\rangle\,[i(-i\sigma_3\overleftarrow{\partial_t} - m)\,V]_\beta\ e^{imt}$$

$$= \langle 0|\,T\,O(t_1)\,\bar{\psi}_\beta(t)\,|0\rangle\ (\sigma_3 V)_\beta\ e^{imt}\Big|_{t=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty} dt\ \langle 0|\,T\,O(t_1)\,\bar{\psi}_\beta(t)|0\rangle\ [i(+i\sigma_3\, im - m)V]_\beta\ e^{imt}$$

$$= \langle 0|\,T\,O(t_1)\,(\bar{\psi}\,\sigma_3 V e^{imt})\,|0\rangle\Big|_{t=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty} \langle 0|\,T\,O(t_1)\,\bar{\psi}_\beta(t)\,|0\rangle\ i\,\cancelto{0}{\left(\begin{pmatrix}-2m&0\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right)_\beta}\ e^{imt}$$

$$= \langle 0|\ T\,O(t_1)\ \underbrace{\psi^{\dagger}_{(t)}V e^{imt}}_{\text{reduces to } b \text{ at } \pm\infty}\ |0\rangle\Big|_{t=-\infty}^{\infty}$$

$$= \langle 0|\ b(+\infty)\,O(t_1) - O(t_1)\,\cancel{b(-\infty)}\,|0\rangle \quad = {}_{out}\langle b|\ O(t_1)\ |0\rangle$$

bosonic

(9) "Pair-creation" amplitude $~~\text{out}~~ \langle ab|0\rangle$.

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; e^{imt_1}\left[-i\bar{u}(i\sigma_3\partial_{t_1}-m)\right]_\alpha \langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle \left[i(-i\sigma_3\overleftarrow{\partial}_{t_2}-m)V\right]_\beta e^{imt_2}$$

$$= \int_{-\infty}^{\infty} dt_1 \left[\left[-i\bar{u}(i\sigma_3\partial_{t_1}-m)\right]_\alpha e^{imt_1}\langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle \; i(-i)(\sigma_3 V)_\beta \, e^{imt_2}\right]_{t_2=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \left\{\left[-i\bar{u}(i\sigma_3\partial_{t_1}-m)\right]_\alpha e^{imt_1}\langle 0|T\psi_\alpha(t_1)\bar{\psi}_\beta(t_2)|0\rangle \; i\underbrace{\left[(+i\sigma_3\, im-m)V\right]_\beta}_{\begin{pmatrix}-2m & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=0} e^{imt_2}\right\}$$

$$= \int_{-\infty}^{\infty} dt_1 \; \left[-i\bar{u}(i\sigma_3\partial_{t_1}-m)\right]_\alpha e^{imt_1}\langle 0|T\psi_\alpha(t_1)\underbrace{(\psi_\beta^\dagger(t_2)V e^{imt_2})}_{\text{reduces to } b(\pm\infty)}|0\rangle\Big|_{t_2=-\infty}^{\infty}$$

$$= \int_{-\infty}^{\infty} dt_1 \; [\quad]_\alpha e^{imt_1}\langle 0|T\psi_\alpha(t_1)b(+\infty) - \psi_\alpha(t_1)\cancel{b(-\infty)|0\rangle}$$

$$= e^{imt_1}(-i\bar{u}\, i\sigma_3)_\alpha \langle 0|T\psi_\alpha(t_1)\, b(+\infty)|0\rangle\Big|_{t_1=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty}\underbrace{\left[-i\bar{u}(-i\sigma_3\, im-m)\right]_\alpha}_{(1\;0)\begin{pmatrix}0 & 0\\ 0 & -2m\end{pmatrix}=0} e^{imt_1}\langle 0|T\psi_\alpha(t_1)b(+\infty)|0\rangle$$

$$= \langle 0|T\underbrace{(u^\dagger\psi_\alpha(t_1)e^{imt_1})}_{\text{reduces to } a(\pm\infty)} b(+\infty)|0\rangle\Big|_{t_1=-\infty}^{\infty}$$

$$= \langle 0|\, a(\infty)b(\infty) - (-1)\, b(\infty)\cancel{a(-\infty)|0\rangle}$$

$$= {}_{\text{out}}\langle ab|0\rangle$$