Feynman parameter integrals

We often deal with products of many propagator factors in loop integrals. The trick is to combine many propagators into a single fraction so that the four-momentum integration can be done easily. This is done commonly using so-called Feynman parameters.

We rewrite the product of propagators

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)\cdots(A_n+i\epsilon)}, \tag{1}$$

where A_i has the form of p^2-m^2. The sign of A_i is not fixed, but the imaginary part has the fixed sign because of $i\epsilon$. This turns out to be useful.

A single propagator can be rewritten using a simple integral

$$\frac{1}{A} = -i\int_0^\infty dt e^{it(A+i\epsilon)}. \tag{2}$$

The surface term at $t\to\infty$ vanishes because of $i\epsilon$ term. Using this multiple times, we find

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)\cdots(A_n+i\epsilon)} = (-i)^n\int_0^\infty dt_1\cdots dt_n e^{i\sum_i^n t_i(A_i+i\epsilon)}. \tag{3}$$

Here comes another trick. We use the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1-\frac{1}{\lambda}\sum_i^n t_i\right). \tag{4}$$

This can be shown using the general formula $\delta(f(x)) = \delta(x-x_0)/|f'(x_0)|$ where x_0 is the zero of $f(x)$, *i.e.*, $f(x_0)=0$. The delta function in the above integral therefore can be rewritten as

$$\delta\left(1-\frac{1}{\lambda}\sum_i^n t_i\right) = \frac{\delta(\lambda-\sum_i^n t_i)}{\frac{1}{\lambda^2}\sum_i^n t_i} = \lambda\delta(\lambda-\sum_i^n t_i), \tag{5}$$

and the integration can be done trivially to yield unity. We used the fact that all of t_i in Eq. (3) are positive to limit the λ integration above zero.

Inserting the unity Eq. (4) into Eq. (3), we find

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)\cdots(A_n+i\epsilon)} = (-i)^n\int_0^\infty dt_1\cdots dt_n\frac{d\lambda}{\lambda}\delta\left(1-\frac{1}{\lambda}\sum_i^n t_i\right)e^{i\sum_i^n t_i(A_i+i\epsilon)}. \tag{6}$$

Now we change the variables from t_i to $t_i=\lambda x_i$, and find

$$= (-i)^n\int_0^\infty dx_1\cdots dx_n d\lambda\lambda^{n-1}\left(1-\sum_i^n x_i\right)e^{i\sum_i^n \lambda t_i(A_i+i\epsilon)}. \tag{7}$$

Now the λ integral can be done by using the integral representation of the gamma function

$$\Gamma(n) = (n-1)! = \int_0^\infty dt\, t^{n-1}e^{-t}. \tag{8}$$

This expression can be generalized to the following one

$$(n-1)!\frac{1}{X^n} = \int_0^\infty dt\, t^{n-1} e^{-tX} \tag{9}$$

as long as $\mathrm{Re}(X) > 0$ so that the integral converges. Because $\mathrm{Re}(-i(A+i\epsilon)) = \epsilon > 0$, we find

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)\cdots(A_n+i\epsilon)} = (-i)^n \int_0^\infty dx_1 \cdots dx_n \frac{(n-1)!}{(-i\sum_i^n t_i(A_i+i\epsilon))^n} \delta\left(1-\sum_i^n x_i\right). \tag{10}$$

Finally, note that all x_i are positive while the sum of x_i must be unity. Therefore the integration region can be limited to $0 < x_i < 1$:

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)\cdots(A_n+i\epsilon)} = (n-1)! \int_0^1 dx_1 \cdots dx_n \frac{1}{(\sum_i^n t_i(A_i+i\epsilon))^n} \delta\left(1-\sum_i^n x_i\right). \tag{11}$$

The simple cases of $n = 2, 3$ are

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)} = \int_0^1 \frac{dx}{(x(A_1+i\epsilon)+(1-x)(A_2+i\epsilon))^2}, \tag{12}$$

$$\frac{1}{(A_1+i\epsilon)(A_2+i\epsilon)(A_3+i\epsilon)} = 2\int_0^1 \frac{dx\, dy\, dz\, \delta(1-x-y-z)}{(x(A_1+i\epsilon)+y(A_2+i\epsilon)+z(A_3+i\epsilon))^3}. \tag{13}$$

General Ideas on the Self-energy Diagrams

We calculate the two-point function $G_2(x-y) = \langle\Omega|\psi(x)\bar{\psi}(y)|\Omega\rangle$ in perturbation theory. An important point is to realize is that the full two-point function can be obtained once one has all the 1PI (one particle irreducible) diagrams $-i\Sigma(\not{p})$ computed because

$$\begin{aligned} G_2(\not{p}) &\equiv \int d^4x G_2(x-y)e^{ip\cdot(x-y)} \\ &= \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}(-i\Sigma(\not{p}))\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}(-i\Sigma(\not{p}))\frac{i}{\not{p}-m_0}(-i\Sigma(\not{p}))\frac{i}{\not{p}-m_0} + \cdots \\ &= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}. \end{aligned} \tag{14}$$

Here, m_0 is the "bare" mass in the Lagrangian which is different from the physical (kinetic) mass.

To correctly identify the mass of the particle, we look for the zero of the denominator in the two-point function. We define the physical mass m of the particle by the equation

$$\not{p}-m_0-\Sigma(\not{p})\big|_{\not{p}=m} = m-m_0-\Sigma(m) = 0. \tag{15}$$

Then we expand the self-energy diagram $\Sigma(\not{p})$ around $\not{p}=m$ as

$$\Sigma(\not{p}) = \delta m - (Z_2^{-1}-1)(\not{p}-m) + Z_2^{-1}\Sigma_R(\not{p}), \tag{16}$$

with $\delta m = \Sigma(m)$, $(Z_2^{-1}-1) = -\partial\Sigma(\not{p})/\partial\not{p}|_{\not{p}=m}$, and $\Sigma_R(\not{p})$ behaves as $O(\not{p}-m)^2$ when $\not{p}\to m$. Then the two-point function becomes

$$G_2(\not{p}) = \frac{i}{\not{p}-m_0-\delta m+(Z_2^{-1}-1)(\not{p}-m_0)-Z_2^{-1}\Sigma_R(\not{p})} = \frac{iZ_2}{\not{p}-m-\Sigma_R(\not{p})}, \tag{17}$$

with $m = m_0+\delta m$. Therefore, δm is interpreted as the correction to the mass of the particle due to interactions, and this is why these diagrams are called "self-energy" diagrams. The factor Z_2 is called the wave-function renormalization factor which describes the strength (square root of the probability) of the field operator ψ creating the one-particle state. At the lowest order in perturbation theory $Z_2=1$, but is different from unity once higher order corrections are taken into account. This is the factor that appears in the LSZ reduction formula.

Explicit Calculation of the One-loop Self-energy Diagram

At the one-loop level, the self-energy diagram is given by

$$-i\Sigma(\not{p}) = (-ie\gamma^\mu)\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{\not{p}+\not{k}-m_0+i\epsilon}\,\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\,(-ie\gamma^\nu). \tag{18}$$

This integral, however, is divergent both in the ultraviolet $k\to\infty$ and the infrared $k\to 0$. To deal with these divergences, we *regulate* the integral, *i.e.*, introduce parameters which make the integral formally convergent, but the parameters must be taken to zero or infinity at the end of calculations. There are many ways of regularizing the integrals, but we adopt the following prescription for this purpose:

$$\frac{1}{k^2+i\epsilon}\to\frac{1}{k^2-\mu^2+i\epsilon}-\frac{1}{k^2-\Lambda^2+i\epsilon}=\frac{\mu^2-\Lambda^2}{(k^2-\mu^2)(k^2-\Lambda^2)}. \tag{19}$$

In the end we take the limit $\mu\to 0$ and $\Lambda\to\infty$ to recover the original expression, but the integral becomes finite as long as we keep both μ and Λ finite. They are called infrared or ultraviolet cutoffs. Below, we often drop $i\epsilon$ terms but it is understood that they are always there.

The regularized form is then

$$-i\Sigma(\not{p}) = -e^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(\not{p}+\not{k}+m_0)\gamma_\mu}{(p+k)^2-m_0^2}\,\frac{\mu^2-\Lambda^2}{(k^2-\mu^2)(k^2-\Lambda^2)}. \tag{20}$$

The next step is to use Feynman parameters to combine three propagator factors and simplify the numerator by the identity $\gamma^\mu\not{a}\gamma_\mu=-2\not{a}$,

$$\begin{aligned}-i\Sigma(\not{p}) &= -e^2 2\int_0^1 dxdydz\delta(1-x-y-z)\\ &\quad\int\frac{d^4k}{(2\pi)^4}\,\frac{(-2(\not{p}+\not{k})+4m_0)(\mu^2-\Lambda^2)}{[x((p+k)^2-m_0^2)+y(k^2-\mu^2)+z(k^2-\Lambda^2)]^3}.\end{aligned} \tag{21}$$

The denominator can be simplified using $x+y+z=1$ to

$$\begin{aligned}\text{denom} &= [k^2+2xk\cdot p+xp^2-xm_0^2-y\mu^2-z\Lambda^2]^3\\ &= [(k+xp)^2+x(1-x)p^2-xm_0^2-y\mu^2-z\Lambda^2]^3.\end{aligned} \tag{22}$$

By shifting the integration variable $k^\mu\to k^\mu-xp^\mu$, we find

$$\begin{aligned}-i\Sigma(\not{p}) &= -e^2 2\int_0^1 dxdydz\delta(1-x-y-z)\\ &\quad\int\frac{d^4k}{(2\pi)^4}\,\frac{(-2((1-x)\not{p}+\not{k})+4m_0)(\mu^2-\Lambda^2)}{[k^2+x(1-x)p^2-xm_0^2-y\mu^2-z\Lambda^2]^3}.\end{aligned} \tag{23}$$

The $\not{k}$ term in the numerator can be dropped because it is an odd function of k^μ and vanishes upon d^4k integration.

The next step is to perform d^4k integration. We need to work out a general formula

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2 + i\epsilon)^3} = \frac{-i}{(4\pi)^2} \frac{1}{2} \frac{1}{M^2 - i\epsilon}. \tag{24}$$

Here we recovered $i\epsilon$ term because it is important. To show this, we first focus on k^0 integration. The (triple) poles are located at $k^0 = \pm\sqrt{\vec{k}^2 + M^2 - i\epsilon}$ and hence the pole with $+$ sign is below the real axis, and that with $-$ sign is above the real axis. Since the integrand goes to zero sufficiently fast at the infinity $|k^0| \to \infty$ on the complex k^0 plane, the integration contour can be rotated from $k^0 \in (-\infty, \infty)$ to $k^0 \in (-i\infty, i\infty)$ without hitting the poles. This is called Wick rotation. Then we can change the integration variable $k^0 = ik^4$ such that the new variable ranges in $(-\infty, \infty)$. Then the above integral is rewritten as

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2 + i\epsilon)^3} = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(-k_E^2 - M^2 + i\epsilon)^3}. \tag{25}$$

Here, $d^4k = dk^0 d\vec{k}$, $d^4k_E = dk^4 d\vec{k}$, and $k_E^2 = (k^4)^2 + \vec{k}^2 = -(k^0)^2 + \vec{k}^2 = -k^2$. Now that the denominator is positive definite (no poles along the integration contour), we do not have to worry about $i\epsilon$ in performing integration. We then use the polar coordinates for four-dimensional k_E^μ and the integration volume is $d^4k_E = 2\pi^2 k_E^3 dk_E = \pi^2 k_E^2 dk_E^2$ after integrating over three angle variables. (In general, $d^nx = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} dx$.) Then the integral becomes quite simple,

$$= \frac{-i\pi^2}{(2\pi)^2} \int_0^\infty dk_E^2 k_E^2 \frac{1}{k_E^2 + M^2 - i\epsilon}. \tag{26}$$

The k_E^2 integration can be done in parts and we find Eq. (24).

Now going back to Eq. (23), we can perform d^4k integration using Eq. (24) and find

$$-i\Sigma(\not{p}) = -e^2 \int_0^1 dxdydz\delta(1-x-y-z) \frac{-i}{(4\pi)^2} \frac{(-2(1-x)\not{p} + 4m_0)(\mu^2 - \Lambda^2)}{xm_0^2 + y\mu^2 + z\Lambda^2 - x(1-x)p^2 - i\epsilon}. \tag{27}$$

The integration over z can be done trivially using the delta function,

$$-i\Sigma(\not{p}) = -e^2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{(-2(1-x)\not{p} + 4m_0)(\mu^2 - \Lambda^2)}{xm_0^2 + y\mu^2 + (1-x-y)\Lambda^2 - x(1-x)p^2 - i\epsilon}. \tag{28}$$

Note that the integration region of y is now limited as $[0, 1-x]$ because of the delta function constraint $x + y + z = 1$ and $z > 0$. The integration over y is also a simple logarithm and we find

$$-i\Sigma(\not{p}) = \frac{-ie^2}{(4\pi)^2} \int_0^1 dx(-2(1-x)\not{p} + 4m_0) \log \frac{xm_0^2 + (1-x)\Lambda^2 - x(1-x)p^2 - i\epsilon}{xm_0^2 + (1-x)\mu^2 - x(1-x)p^2 - i\epsilon}. \tag{29}$$

Even the integration over x can be done with elementary functions only, but the expression becomes lengthy and not very inspiring and therefore we keep x integration. Finally using $\alpha = e^2/4\pi$,

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi}\int_0^1 dx(-2(1-x)\not{p} + 4m_0)\log\frac{xm_0^2 + (1-x)\Lambda^2 - x(1-x)p^2 - i\epsilon}{xm_0^2 + (1-x)\mu^2 - x(1-x)p^2 - i\epsilon}. \tag{30}$$

Following the general discussions, we now identify δm and Z_2. Since $\Sigma(\not{p})$ and hence $\delta m = m - m_0$ are $O(e^2)$, I can replace m_0 in $\Sigma(\not{p})$ by m by neglecting $O(e^4)$ corrections.

First, $\delta m = \Sigma(\not{p})|_{\not{p}=m}$ is given by

$$\begin{aligned}\delta m &= \Sigma(\not{p})|_{\not{p}=m} = \frac{\alpha}{4\pi}\int_0^1 dx(-2(1-x)m + 4m)\log\frac{xm^2 + (1-x)\Lambda^2 - x(1-x)m^2 - i\epsilon}{xm^2 + (1-x)\mu^2 - x(1-x)m^2 - i\epsilon} \\ &= m\frac{\alpha}{4\pi}\int_0^1 dx 2(1+x)\log\frac{x^2m^2 + (1-x)\Lambda^2 - i\epsilon}{x^2m^2 + (1-x)\mu^2 - i\epsilon}. \end{aligned}\tag{31}$$

The argument of the logarithm is manifestly positive, and we can safely drop $i\epsilon$. Moreover, we take the limit $\Lambda \to \infty$ and $\mu \to 0$ in the end, and we can neglect x^2m^2 in the numerator and $(1-x)\mu^2$ in the denominator. Then the expression becomes drastically simpler and the end result is

$$\delta m = m\frac{\alpha}{4\pi}\left[3\log\frac{\Lambda^2}{m^2} + \frac{3}{2}\right]. \tag{32}$$

This is the correction to the mass of the electron.

The interpretation of this self-energy is quite interesting. In classical electrodynamics, we actually have a *linearly-divergent* self-energy. An electron creates a Coulomb field around it, and it feels its own Coulomb field. If, for instance, one imagines the electron to be a sphere of radius r_e with a uniform charge density, the total potential energy is $V = \frac{3}{5}\frac{e^2}{r_e}$. The total energy of the electron is the sum of the rest energy m_0c^2 and the potential energy V and hence the "total" mass of the electron we observe is given by

$$mc^2 = m_0c^2 + \frac{3}{5}\frac{e^2}{4\pi r_e}. \tag{33}$$

In the limit of $r_e \to 0$, the bare mass m_0 needs to be sent *negative* to cancel the linearly divergent Coulomb self-energy to obtain the observed mass of the electron. In the quantum mechanical language, cleary this is an "ultraviolet" divergence as it corresponds to short-distance physics $r_e \to 0$. If we imagine the electron to be as small as the Planck size $r_e = \sqrt{\hbar G_N/c^3} = 1.6 \times 10^{-33}$ cm, where presumably the quantum gravity takes over physics, we need to make the bare mass as negative as -5.34×10^{19} MeV which is cancelled by the self-energy for 20 digits to get 0.511 MeV. This is absurd. The classical electrodynamics therefore breaks down at the distance scale where the rest energy and the self-energy become comparable, $r_e \simeq e^2/mc^2 \simeq 10^{-13}$ cm, and a better (deeper) theory needs to take over the classical electrodynamics below this distance scale.

What we have learnt here is that the situation in the QED is much better. The total mass of the electron is

$$mc^2 = m_0c^2\left(1+\frac{\alpha}{4\pi}\left[3\log\frac{\Lambda^2}{m^2}+\frac{3}{2}\right]\right). \tag{34}$$

The ultraviolet cutoff Λ corresponds to the inverse "size" of the electron in the classical language. Again if we imagine the electron to be as small as the Planck size, the correction to the electron mass is only 18.0%. The crucial difference from the classical theory is that (1) the dependence on the "size" of the electron is only logarithmic instead of power, and (2) the correction is proportional to the electron mass itself and hence can never be much larger than the bare mass. Compared to the classical case Eq. (33), the quantum result Eq. (34) corresponds to the cutoff $r_e \simeq \hbar/mc$ which is nothing but the Compton wave length. Below this distance scale, quantum effects are essential and the classical theory does not apply any more.

There are several lessons to be learnt from this discussion. First, the QED knows its own limitation by the fact that it requires the ultraviolet cutoff to regulate the theory. The theory does not apply beyond certain energy scale. Such a theory is called an "effective field theory," and is true to almost all quantum field theories. However, the cutoff can be extremely large unlike in the classical electrodynamics. Even though the QED should be regarded as a theory which should be taken over by a yet deeper theory at extreme high energies, its applicability is practically infinite. Second, even though there is an ultraviolet divergence in the electron mass, what we observe is the *total* of the bare mass and the self energy. Therefore, any calculations should be expressed in terms of the *observed* mass of the electron instead of the bare mass. The same comment applies to the fine-structure constant as we will see later. It turns out that any physical quantities are finite once expressed in terms of the *observed* mass and fine-structure constants despite the fact that there are dependence on the ultraviolet cutoff as well as bare parameters at the intermediate stage of calculations. This is a general property of *renormalizable* quantum field theories: all physical quantities are finite once expressed in terms of observable quantities. Because of this property, one can use the QED to do precise calculations without worrying about what physics is there at the energy scale of the ultraviolet cutoff.

Next topic is the wave-function renormalization factor. Following the general discussions, we calculate

$$Z_2^{-1} - 1 = -\left.\frac{\partial\Sigma(\not{p})}{\partial\not{p}}\right|_{\not{p}=m}. \tag{35}$$

Using Eq. (30) again, and paying attention to the fact that $p^2 = (\not{p})^2$, we find

$$\begin{aligned} Z_2^{-1} - 1 &= -\frac{\alpha}{4\pi}\int_0^1 dx\left[-2(1-x)\log\frac{xm^2+(1-x)\Lambda^2-x(1-x)p^2-i\epsilon}{xm^2+(1-x)\mu^2-x(1-x)p^2-i\epsilon}\right. \\ &\quad \left.+(-2(1-x)\not{p}+4m)\left(\frac{-x(1-x)2\not{p}}{xm^2+(1-x)\Lambda^2-x(1-x)p^2-i\epsilon}\right.\right. \end{aligned}$$

$$-\frac{-x(1-x)2\not{p}}{xm^2+(1-x)\mu^2-x(1-x)p^2-i\epsilon}\Bigg)\Bigg]\Bigg|_{\not{p}=m}. \tag{36}$$

We can drop the first term in the parenthesis because it vanishes in $\Lambda \to 0$ limit, and also m^2 (μ^2) in the numerator (denominator) in the logarithm can be set zero. After a simple integration, we find

$$Z_2^{-1} - 1 = -\frac{\alpha}{4\pi}\left[2\log\frac{m^2}{\mu^2} - \log\frac{\Lambda^2}{m^2} - \frac{9}{2}\right]. \tag{37}$$