HW #9, due Mar 24

1. Abelian Sigma Model The Abelian Sigma Model is a model of spontaneous $U(1)$ symmetry breaking. The Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi)^* \partial^\mu \phi - V(\phi), \tag{1}$$

where the potential term is

$$V(\phi) = -\mu^2 |\phi|^2 + \lambda |\phi|^4. \tag{2}$$

(a) Minimize the potential with respect to ϕ (treat ϕ and ϕ^* as independent variables) and solve for the vacuum expectation value of $\phi_0 = \langle \phi \rangle$.

(b) We expand the field ϕ around its minimum as follows:

$$\phi = \left(\phi_0 + \frac{\sigma}{\sqrt{2}} \right) e^{i\chi/\sqrt{2}\phi_0}. \tag{3}$$

The field χ is the Nambu–Goldstone boson. Write down the entire Lagrangian density in terms of σ and χ and identify their masses.

2. Abelian Higgs Model The Abelian Higgs Model is a $U(1)$ gauge theory coupled to a Klein–Gordon field with a symmetry breaking potential. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi), \tag{4}$$

with $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$ and the same potential term as in the Abelian Sigma Model.

(a) We expand the field ϕ around its minimum as follows:

$$\phi = \left(\phi_0 + \frac{H}{\sqrt{2}} \right) e^{i\chi/\sqrt{2}\phi_0}. \tag{5}$$

Unlike the case of the global (ungauged) $U(1)$ symmetry, we can always do a gauge transformation of the field ϕ to make the Nambu–Goldstone degrees of freedom χ disappear. This particular choice of gauge is called the "unitary gauge." Write down the entire Lagrangian density in terms of the gauge field A_μ and the "Higgs field" H and identify their masses.

(b) Show that the propagator of the gauge field is given by

$$\frac{i}{q^2 - m^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2} \right). \tag{6}$$