229C HW # 2 (due Oct 1, 12:30)

1. To gain intuitions on the neutrino oscillation effects in the atmospheric neutrino data, make (nine) plots of the $\cos\theta_z$ distributions in 5 bins, for three neutrino energies $E_\nu = 200$ MeV, 2 GeV, and 20 GeV, and for three mass-squared differences $\Delta m^2 c^4 = 3 \times 10^{-4}$ eV^2, 3×10^{-3} eV^2, and 3×10^{-2} eV^2. Assume a completely isotropic distribution of neutrinos (as opposed to the true distribution peaked towards the horizontal direction $\cos\theta_z = 0$). Take the maximal mixing $\sin^2 2\theta = 1$, and assume the fixed height $h = 20$ km where neutrinos are produced in the atmosphere.

2. Let us derive a formula for the neutrino oscillation probabilities in the case of three generations. In general, the neutrino mass-squared matrix is given (in the basis where the charged lepton mass is diagonalized) by

$$\mathcal{M}_\nu^2 = U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger, \qquad U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}, \tag{1}$$

where m_i^2 are three real eigenvalues and U is the unitary diagonalization matrix.

(1) Follow the same analysis as done in the class to show that the neutrino oscillation probabilities are given by

$$P(\nu_\mu \to \nu_e) = \sum_{i,j=1}^{3} U_{ei} U_{ej}^* U_{\mu i}^* U_{\mu j} \exp\left(-i \frac{(m_i^2 - m_j^2) c^4 L}{2\hbar c E} \right). \tag{2}$$

and similarly for other combinations when $E \gg m_i^2$.

(2) Check that the above formula reduces to the two-generation formula when $U_{e3} = U_{\mu 3} = U_{\tau 1} = U_{\tau 2} = 0$, $U_{e1} = U_{\mu 2} = \cos\theta$ and $U_{e2} = -U_{\mu 1} = \sin\theta$. Show that the probabilities are real (in the sense that it does not have an imaginary part). Also check that the total probability is one: $P(\nu_\mu \to \nu_e \text{ or } \nu_\mu \text{ or } \nu_\tau) = 1$.

(3) Show that there is a possible time reversal violation $P(\nu_\mu \to \nu_e) \neq P(\nu_e \to \nu_\mu)$ if the unitary matrix U is not real.

(4) For anti-neutrinos, the mass-squared matrix is complex conjugated: $\mathcal{M}_{\bar\nu}^2 = \mathcal{M}_\nu^{2*}$. Obtain the oscillation probabilities for anti-neutrinos and show that there is a possible CP violation $P(\bar\nu_\mu \to \bar\nu_e) \neq P(\nu_\mu \to \nu_e)$.

(5) Show that CPT is always preserved: $P(\nu_\mu \to \nu_e) = P(\bar\nu_e \to \bar\nu_\mu)$.