Hydrostatic Equilibrium with Degenerate Fermions, Type-II Supernova, and Neutrino Burst

Because of the lack of time, I went through this discussion very quickly. In case that you coudn't write down all the equations, and to make the discussion more complete, I'll repeat it in more details in this note.

You start from the hydrostatic equilibrium

$$-\frac{dp}{dr} = \frac{GM(r)}{r^2}\rho(r), \tag{1}$$

where p is the pressure, G the Newton's constant, $\rho(r)$ the mass density at the radius r, and

$$M(r) = \int_0^r 4\pi r'^2 dr' \rho(r) \tag{2}$$

is the total mass inside the radius r. The spherical symmetry is assumed.

By combining the above two equations, I obtain a second-order differential equation

$$\frac{d}{dr}\frac{r^2}{\rho(r)}\frac{dp}{dr} = -4\pi G r^2 \rho(r). \tag{3}$$

The unknown functions here are $\rho(r)$ and $p(r)$.

We assume an equation of state

$$p(r) = K\rho(r)^\gamma, \tag{4}$$

where K, γ are constants. This is not true always, but is in certain limits we will consider later. For specific type of gas (or whatever), we have different K and γ. This is where microscopic physics comes in. We continue the discussions at the level of macroscopic physics with K and γ taken as some constants, and come back to their specific values for the case of degenerate fermions later.

The above second-order differential equation can be solved semi-analytically, using the following changes in the variables:

$$r = \left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{1/2} \rho(0)^{(\gamma-2)/2}\xi, \tag{5}$$

$$\rho = \rho(0)\theta^{1/(\gamma-1)}, \tag{6}$$

$$p = K\rho(0)^\gamma \theta^{\gamma/(\gamma-1)}. \tag{7}$$

After simplifying the equations, we obtain differential equations for dimensionless quantities ξ and $\theta(\xi)$:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi}+\theta^{1/(\gamma-1)}=0, \tag{8}$$

with the boundary conditions $\theta(0)=1$ (normalization) and $\theta'(0)=0$. The latter boundary condition comes from analyticity (or absence of singularity) at the origin. Close to the origin, $M(r)\approx\frac{4\pi}{3}\rho(0)r^3$, and the hydrostatic equilibrium requires $-p'(r)\approx\frac{4\pi}{3}\rho(0)r$. Therefore, $p'(r)$ is linear in r and hence $p'(0)=0$. This in turn together with the equation of state $p=K\rho^\gamma$ implies $\rho'(0)=0$. Therefore $\theta'(0)=0$ as well. The function $\theta(\xi)$ is known as Lane-Emden function of index $1/(\gamma-1)$. The function is plotted in Fig. 1 for $\gamma=4/3$ (wider) and 5/3 (more compact).

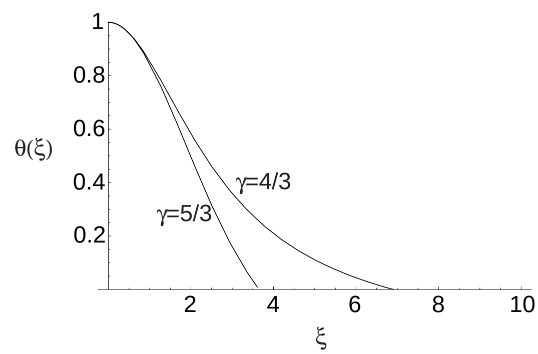

Figure 1: Plot of Lane-Emden function $\theta(\xi)$ for $\gamma=4/3$ and 5/3.

The function $\theta(\xi)$ has a zero at a finite value of the argument ξ if $\gamma>6/5$ (we'll be interested in the case $4/3\le\gamma\le5/3$ later). We call ξ at the zero as ξ_1. They are approximately $\xi_1=6.90$ for $\gamma=4/3$ and $\xi_1=3.65$ for $\gamma=5/3$. The radius of the core is given simply by ξ_1:

$$R=\left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{1/2}\rho(0)^{(\gamma-2)/2}\xi_1, \tag{9}$$

where $\rho(0)$ is the only free parameter for fixed K and γ. Similarly, the total mass of the core is given by

$$\begin{aligned} M &= \int_0^R 4\pi r^2\rho(r)dr \\ &= 4\pi\rho(0)^{(3\gamma-4)/2}\left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{1/2}\int_0^{\xi_1}\xi^2\theta(\xi)^{1/(\gamma-1)}d\xi. \end{aligned} \tag{10}$$

By using the differential equation, the integrand is the same as $-d(\xi^2\theta')/d\xi$ and hence can be integrated:

$$M = 4\pi\rho(0)^{(3\gamma-4)/2}\left[\frac{K\gamma}{4\pi G(\gamma-1)}\right]^{1/2}\xi_1^2|\theta'(\xi_1)|. \tag{11}$$

The values of $\xi_1^2|\theta'(\xi_1)|$ are 2.02 and 2.71 for $\gamma = 4/3$, $5/3$, respectively.

An interesting point in the above solutions is that the total mass of the core M is proportional to $R^{-(3\gamma-4)/(2-\gamma)}$ after eliminating $\rho(0)$ but for fixed K and γ, and the core is smaller for more massive core for $4/3 < \gamma < 2$. And the mass does not depend on the center density $\rho(0)$ at all for $\gamma \to 4/3$.

Now we turn to microscopic physics of degenerate fermions. First let us consider non-relativistic electrons. With the Fermi-momentum k_F, the number density is given simply by

$$n = 2\int^{k_F}\frac{d^3k}{(2\pi\hbar)^3} = \frac{k_F^3}{3\pi^2\hbar^3}. \tag{12}$$

The total mass density is dominated by the accompanying nuclei (^{56}Fe for a massive star just before its collapse), and is given by

$$\rho = n m_N \mu, \tag{13}$$

where m_N is the mass of the nucleon (difference between protons, nucleons, and bound nucleons ignored; error of the order of 8 MeV) and μ is the number of nucleons per electron ($\mu = 56/26 = 2.15$ for ^{56}Fe). The non-relativistic approximation is valid as long as the Fermi momentum is much smaller than $m_e c$, or if

$$\rho \ll \rho_c = m_N\mu\frac{m_e^3c^3}{3\pi^2\hbar^3}. \tag{14}$$

The energy density of electrons is given by their kinetic energies

$$\varepsilon = 2\int^{k_F}\frac{d^3k}{(2\pi\hbar)^3}\frac{k^2}{2m_e} = \frac{2}{(2\pi\hbar)^3 2m_e}\frac{4\pi}{5}k_F^5. \tag{15}$$

In order to find K and γ, we need to obtain an expression for the pressure p. This is done using the first law of thermodynamics, $dU = -pdV$ under an infinitesimal change of the volume. Using the number density $n = N/V$ for the total number of electrons N in a volume V, we solve for k_F:

$$k_F = \left(3\pi^2\hbar^3\frac{N}{V}\right)^{1/3}, \tag{16}$$

and we then find

$$U = \varepsilon V \propto V^{-2/3}. \tag{17}$$

From the power, we immediately find

$$dU = -\frac{2}{3}\frac{U}{V}dV = -\frac{2}{3}\varepsilon dV, \tag{18}$$

and hence

$$p = \frac{2}{3}\varepsilon = \frac{1}{15\pi^2\hbar^3 m_e}\left(3\pi^2\hbar^3\frac{\rho}{m_N\mu}\right)^{5/3}. \tag{19}$$

We find $\gamma = 5/3$ from the power in ρ, and the rest gives the constant K.

We now find the radius R and the mass M of the degenerate core. Using the previous formulae,

$$\begin{aligned} M &= 4\pi\rho(0)^{1/2}\left[\frac{1}{15\pi^2\hbar^3 m_e}\left(\frac{3\pi^2\hbar^3}{m_N\mu}\right)^{5/3}\frac{1}{4\pi G}\frac{5}{2}\right]^{3/2} \times 2.71 \\ &= \left(\frac{\rho(0)}{\rho_c}\right)^{1/2}\mu^{-2}\left(\frac{3\pi\hbar^3 c^3}{32G^3 m_N^4}\right)^{1/2} \times 2.71 \\ &= 2.73\mu^{-2}\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot, \end{aligned} \tag{20}$$

where $M_\odot$ is the solar mass. It is quite amazing the the combination of fundamental constants gives the correct mass scale for a star. Within this non-relativistic approximation, the total mass can be increased by increasing the central density $\rho(0)$, while the approximation breaks down when $\rho(0)$ approaches ρ_c. The radius is given by

$$\begin{aligned} R &= \left[\frac{1}{15\pi^2\hbar^3 m_e}\left(\frac{3\pi^2\hbar^3}{m_N\mu}\right)^{5/3}\frac{1}{4\pi G}\frac{5}{2}\right]^{1/2}\rho(0)^{-1/6} \times 3.65 \\ &= \left(\frac{\rho_c}{\rho(0)}\right)^{1/6}\mu^{-1}\left[\frac{3\pi}{8}\frac{\hbar^3}{Gm_N^2 m_e^2 c}\right]^{1/2} \times 3.65 \\ &= 2.0 \times 10^4\ \text{km}\left(\frac{\rho_c}{\rho(0)}\right)^{1/6}\mu^{-1}. \end{aligned} \tag{21}$$

As noted earlier, the radius is smaller for the larger mass.

Once the mass of the core is larger, we should abandon non-relativistic approximation. Let us assume the extreme limit that all electrons are relativistic. The only change from the previous analysis is that the kinetic energy is kc rather than $k^2/2m_e$. This changes the energy density to

$$\varepsilon = 2\int^{k_F} \frac{d^3k}{(2\pi\hbar)^3} ck = \frac{2\pi c}{(2\pi\hbar)^3} k_F^4. \tag{22}$$

As a result, the energy depends on the volume as $V^{-1/3}$ and hence

$$p = \frac{1}{3}\varepsilon = \frac{c}{12\pi^2\hbar^3}\left(3\pi^2\hbar^3\frac{\rho}{m_N\mu}\right)^{4/3}. \tag{23}$$

We find $\gamma = 4/3$ from the dependence on ρ and the rest gives the constant K. The formula for the mass is now changed to

$$\begin{aligned} M &= 4\pi\rho(0)^0\left[\frac{c}{12\pi^2\hbar^3}\left(\frac{3\pi^2\hbar^3}{m_N\mu}\right)^{4/3}\frac{4}{4\pi G}\right]^{3/2} \times 2.02 \\ &= \left(\frac{3\pi}{4}\frac{\hbar^3c^3}{G^3m_N^4}\right)^{1/2}\mu^{-2} \\ &= 5.76\mu^{-2}M_\odot. \end{aligned} \tag{24}$$

The result does not depend on the central density $\rho(0)$. Therefore, even when one makes the center of the core arbitrarily dense, the total mass cannot exceed roughly $1.2M_\odot$ for $\mu \sim 2.15$. The radius, on the other hand, does depend on $\rho(0)$, and is given by

$$\begin{aligned} R &= \frac{1}{2}\left(\frac{3\pi\hbar^3}{cGm_e^2m_N^2\mu^2}\right)^{1/2}\left(\frac{\rho_c}{\rho(0)}\right)^{1/3} \times 6.90 \\ &= 5.3 \times 10^4 \text{ km}\left(\frac{\rho_c}{\rho(0)}\right)^{1/3}\mu^{-1}. \end{aligned} \tag{25}$$

The radius shrinks as $\rho(0)^{-3}$ which keeps the total mass constant as one increases $\rho(0)$.

The bottom line of this analysis is that one cannot support a core more massive than $1.2M_\odot$ by electron degeneracy pressure and the core collapses further in this case. Note that once the density $\rho(0)$ is large enough, the

Fermi energy E_F can be above the threshold for the reaction $e^- p \to \nu_e n$: $(m_n^2 - m_p^2 - m_e^2)c^2/2m_p = 1.294$ MeV. This happens when $\rho_c \gtrsim 16\rho_c$. Once this is the case, the electrons get more and more captured by protons and are converted to neutrons, which results in a loss of pressure and the collapse proceeds further. Eventually all the electrons are absorbed by protons and the entire core will consist of neutrons.[1]

After the further collapse, the mass will be supported by the degeneracy pressure of neutrons. The analysis is the same as the case of degenerate electrons except that we replace m_e by m_N and set $\mu = 1$ because we have one nucleon (neutron) per one neutron. Let us assume the non-relativistic case. Then we find

$$M = 2.75 \left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot, \tag{26}$$

$$R = 10 \text{ km} \left(\frac{\rho_c}{\rho(0)}\right)^{1/6}. \tag{27}$$

Note that ρ_c in these formulae are that of neutron: $\rho_c = \frac{m_N^2 c^3}{3\pi^2\hbar^3} = 6.11 \times 10^{15}$ g/cm^3. This is an incredibly high density. If we assume that all of the mass of the degenerate electron core of $M \gtrsim 1.2 M_\odot$ is kept in the degenerate neutron core, $\rho(0) \gtrsim 0.19\rho_c$. For larger M, even neutrons become relativistic and eventually cannot be supported by the neutron degeneracy pressure, and the core collapses to a black hole. For this analysis, however, one should take the nuclear "hard-core" repulsive force among neutrons into account as well as general relativistic corrections, which are beyond the scope of this note.

For a (barely) non-relativistic neutron core $\rho(0) \sim 0.2\rho_c$, the core collapse should release the total gravitational self-energy in either radiation, neutrinos, or gravitational waves:

$$\Delta E \simeq \frac{GM^2}{R} = 2.0 \times 10^{54} \text{ erg} \left(\frac{\rho(0)}{\rho_c}\right)^{5/6} \sim 5 \times 10^{53} \text{ erg}. \tag{28}$$

On the other hand, observations of Type II supernova[2] suggest the total luminosity in radiation of 10^{49} erg and in kinetic energy of 10^{51} erg. The

[1] This is not completely correct because of the chemical equilibrium $\nu_e n \leftrightarrow pe$. But this doesn't change any of the order-of-magnitude results later. More serious calculations include this equilibrium together with pion and kaon degrees of freedom.

[2] Type II supernovae have hydrogen lines while Type I supernovae don't. They are

simulations suggest the total energy in gravitational wave is also of the order of 10^{51} erg. The rest must be emitted in the form of neutrinos.

Note that the mean free path of the neutrinos in the neutron star core is much shorter than the radius. The average neutrino cross section with a temperature T is given by

$$\sigma \sim 4\frac{G_F^2}{\pi}T^2. \tag{29}$$

With $T \sim 10$ MeV, as expected from nuclear equilibrium (not explained here in this note), the average mean free path can be estimated as

$$\lambda_\nu = 24 \text{ m} \left(\frac{T}{\text{MeV}}\right)^{-2} \left(\frac{\rho}{10^{14}\text{g/cm}^3}\right)^{-1}. \tag{30}$$

Therefore the neutrinos can only diffuse out; cannot escape directly from their birth points. This fact is usually referred to as "trapping" of neutrinos. The neutrino emission can therefore be approximated by a black-body radiation from the surface. The emission rate is then given by Stefan-Boltzmann law:

$$L = \frac{7}{8}\sigma T^4 4\pi R^2 = 1.1 \times 10^{53} \text{ ergsec}^{-1} \left(\frac{T}{10 \text{ MeV}}\right)^4 \left(\frac{R}{10 \text{ km}}\right)^2 \tag{31}$$

for one neutrino species (*e.g.* ν_μ and $\bar{\nu}_\mu$). The factor of 7/8 comes from the fact that the black-body radiation law is slighly different for fermions. Therefore the neutrino esmission for the surface can completely account for the required release of the total gravitational energy with the expected temperature with a burst of a few seconds.

Now comes the comparison to the observation. Assuming an exponentially decaying surface temperature $T = T_0 e^{-t/4\tau}$, Spergel *et al* (Science **237**, 1471 (1987)) has found the best fit to the Kamiokande and IMB observation of neutrino events with $T_0 = 4.2^{+1.2}_{-0.8}$ MeV, $\tau = 4.5^{+1.7}_{-2.0}$ sec and the total luminosity in $\bar{\nu}_e$:

$$\int L_{\bar{\nu}_e} dt = 6.1^{+3.5}_{-3.6} \times 10^{52} \text{ erg}. \tag{32}$$

The reason for specifying $\bar{\nu}_e$ is because all the neutrino events seen by Kamiokande and IMB are probably those with the reaction $\bar{\nu}_e p \to e^+ n$ which has the

either caused by collapse of stars with hydrogen in the envelope somehow blown off earlier (Ib and Ic), or by completely different mechanism such as destruction of white dwarfs in binary systems (Ia) and hence has nothing to do with this analysis.

largest cross section at this energy (cf. HW #1). The total neutrino luminosity is roughly 6 times larger, which is in a good agreement with the above-estimated gravitational energy release.

Dynamics of the burst and supernova explosion itself is much more complicated than energetics discussed here. The point is that further neutrino measurements of a galactic supernova will map out the time profile and energy distribution of neutrinos (with order 6000 neutrinos at SuperK) together with a study of its composition (with order 400 neutral current events at SNO). This would settle many of the on-going debates on the dynamics of the explosion. It is amusing that supernova explosion occurs a few hours or even days after the core collapse (*i.e.*, neutrino burst). Since solar neutrino detectors continuously monitor the entire 4π solid angle, they cannot miss a galactic supernova.