Thermodynamics

In Einstein equation, the energy momentum tensor of homogeneous, isotropic fluid takes the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}, \tag{1}$$

and we need to know the energy density ρ and the pressure p for different kinds of fluids. Here and hereafter, we use the natural unit $c = \hbar = k = 1$, where k is the Boltzman constant.

1 Bosons

For the superrelativistic gas of free bosons ($T \gg m$), we ignore the mass $m \to 0$ and use $E_k = k$,

$$\begin{aligned} \rho &= g \int \frac{d^3k}{(2\pi)^3} \frac{E_k}{e^{E_k/T} - 1} \\ &= g \frac{4\pi}{(2\pi)^3} \int_0^\infty dk k^2 \frac{k}{e^{k/T} - 1} \\ &= g \frac{1}{2\pi^2} \int_0^\infty dk k^3 \sum_{n=1}^\infty e^{-nk/T} \\ &= g \frac{1}{2\pi^2} \Gamma(4) \sum_{n=1}^\infty \left(\frac{T}{n}\right)^4 \\ &= g \frac{1}{2\pi^2} 3! T^4 \zeta(4) \\ &= g \frac{\pi^2}{30} T^4. \end{aligned} \tag{2}$$

Here, we used a special value of the ζ function, $\zeta(s) = \sum_{n=1} n^{-s}$ and $\zeta(4) = \pi^4/90$. The spin multiplicity factor is $g = 2s + 1$ for massive particles ($g = 2$ for electrons), but $g = 2$ for photons and $g = 1$ for neutrinos.

The entropy can be calculated using the 1st law of thermodynamics, $T = (\partial U/\partial S)_V$. Since $U = \rho V$ is expressed in terms of the temperature above, we find $T = (\partial U/\partial T)_V \times (dT/dS) = g\frac{\pi^2}{30} 4T^3 (dT/dS)$, and hence

$$\frac{dS}{dT} = g \frac{\pi^2}{30} 4T^2 V. \tag{3}$$

Solving this simple differential equation, we obtain the entropy density $s = S/V$,

$$s = g\frac{2\pi^2}{45}T^3. \tag{4}$$

The entropy is an important quantity because the Universe expands without any supply of heat (energy) from "outside." Therefore the total entropy is conserved (adiabatic process).

Finally, the pressure of the gas is obtained from the free energy $F = U - TS = -g(\pi^2/90)T^4V$ and its derivative

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = g\frac{\pi^2}{90}T^4 = \frac{1}{3}\rho. \tag{5}$$

The number density can be calculated similarly to the energy density:

$$\begin{aligned}
n &= g\int\frac{d^3k}{(2\pi)^3}\frac{E_k}{e^{E_k/T}-1} \\
&= g\frac{4\pi}{(2\pi)^3}\int_0^\infty dk k^2\frac{k}{e^{k/T}-1} \\
&= g\frac{1}{2\pi^2}\int_0^\infty dk k^2\sum_{n=1}^{\infty}e^{-nk/T} \\
&= g\frac{1}{2\pi^2}\Gamma(3)\sum_{n=1}^{\infty}\left(\frac{T}{n}\right)^3 \\
&= g\frac{1}{2\pi^2}2!T^3\zeta(3) \\
&= g\frac{\zeta(3)}{\pi^2}T^3. \qquad (6)
\end{aligned}$$

Note that $\zeta(3) = 1.20206\ldots$

2 Fermions

We basically repeat the same calculation with the Fermi-Dirac distribution

$$\begin{aligned}
\rho &= g\int\frac{d^3k}{(2\pi)^3}\frac{E_k}{e^{E_k/T}+1} \\
&= g\frac{4\pi}{(2\pi)^3}\int_0^\infty dk k^2\frac{k}{e^{k/T}+1}
\end{aligned}$$

$$
\begin{aligned}
&= g\frac{1}{2\pi^2}\int_0^\infty dk k^3 \sum_{n=1}^{\infty}(-1)^{n-1}e^{-nk/T} \\
&= g\frac{1}{2\pi^2}\Gamma(4)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{T}{n}\right)^4 . \qquad (7)
\end{aligned}
$$

The trick is to express the sum $\sum_{n=1}(-1)^{n-1}n^{-4}$ in terms of $\zeta(4)$. By dividing the sum into odd and even numbers,

$$
\sum_{n=1}\frac{(-1)^{n-1}}{n^4} = \sum_{\text{odd}}\frac{1}{n^4} - \sum_{\text{even}}\frac{1}{n^4} = \sum_{n=1}\frac{1}{n^4} - 2\sum_{\text{even}}\frac{1}{n^4}. \qquad (8)
$$

Rewriting the sum over even numbers with $n = 2m$, we find $\sum_{\text{even}} = \sum_{m=1}(2m)^{-4} = \sum_{m=1} m^{-4}/16 = \zeta(4)/16$. Therefore,

$$
\sum_{n=1}\frac{(-1)^{n-1}}{n^4} = \zeta(4) - 2\frac{\zeta(4)}{16} = \frac{7}{8}\zeta(4). \qquad (9)
$$

The only change from the boson case is that ρ, s, and p are all multiplied by the same factor 7/8:

$$
\begin{aligned}
s &= \frac{7}{8}g\frac{2\pi^2}{45}T^3, \qquad (10)\\
p &= \frac{1}{3}\rho. \qquad (11)
\end{aligned}
$$

The number density now depends on the sum $\sum_{n=1}(-1)^{n-1}n^{-3}$. Following the same trick as above, we find $\sum_{n=1}(-1)^{n-1}n^{-3} = (3/4)\zeta(3)$. Therefore,

$$
n = \frac{3}{4}g\frac{\zeta(3)}{\pi^2}T^3. \qquad (12)
$$

3 Total

By adding constributions of bosons and fermions, we find

$$
\begin{aligned}
\rho &= g_*\frac{\pi^2}{30}T^4, \qquad (13)\\
s &= g_*\frac{2\pi^2}{45}T^3, \qquad (14)\\
p &= g_*\frac{1}{3}\rho, \qquad (15)
\end{aligned}
$$

with an effective total degrees of freedom

$$g_* = \sum_{\text{bosons}} g_{\text{bosons}} + \frac{7}{8} \sum_{\text{fermions}} g_{\text{fermions}}. \tag{16}$$

For the number density, we instead find

$$n = g'_* \frac{\zeta(3)}{\pi^2} T^3 \tag{17}$$

with

$$g'_* = \sum_{\text{bosons}} g_{\text{bosons}} + \frac{3}{4} \sum_{\text{fermions}} g_{\text{fermions}}. \tag{18}$$