HW #1

1. Age of the Universe

(a)

Because of the assumption that the universe is flat, $k = 0$, and the energy density scales as $\rho \propto R^{-3}$ for matter-dominated universe. We use the convention that the scale factor now $R(t_0) = 1$, possible because there is no physical meaning to the scalar factor for the flat geometry. (In the case of the closed geometry, the scale factor has the meaning of the size of the uinverse, etc.) Hence, $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G_N \rho = \frac{8\pi}{3} G_N \rho_0 R^{-3}$. Because this equation also holds at $t = t_0$ (now), where the l.h.s. of the equation is nothing but the current Hubble "constant" $H_0 = \frac{\dot{R}}{R}(t_0)$, we find $\frac{8\pi}{3} G_N \rho_0 = H_0{}^2$. Therefore the Friedmann equation becomes $\left(\frac{\dot{R}}{R}\right)^2 = H_0{}^2 R^{-3}$, and hence $R \dot{R}^2 = H_0{}^2$. The equation to integrate then is $R^{1/2}\, dR = H_0\, dt$, and we find $\frac{2}{3} R^{3/2}(t) - R^{1/2}(0) = H_0\, t$. Because of the convention $R(t_0) = 1$, and the initial condition $R(0) = 0$ (Big Bang singularity), we find $t_0 = \frac{2}{3} H_0{}^{-1}$. $t_0 = \frac{2}{3}\left(71\,\text{km sec}^{-1}\,\text{Mpc}^{-1}\right)^{-1} = 2\left(71\,\text{km sec}^{-1}(3.26210^6\,\text{lyr})^{-1}\right)^{-1} = 9.19\,\text{Gyr}$ about nine billion years. Compared to the age of old stars estimated to be 13.6±0.8 Gyr, the universe is too young!

$$\frac{2}{3}\left(\frac{71}{3.00\ 10^5\ 3.262\ 10^6}\right)^{-1}$$

$$9.18873 \times 10^9$$

(b)

If there is also the curvature contribution to the current universe, there is one more free parameter to derive the age of the universe. Let us first consider the extreme case of empty universe (no matter) but the curvature alone. This is possible only for the open universe. In this case, $\left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{R^2}$, and $\dot{R} = 1$. The solution is simply $R(t) = t$. On the other hand, the current values of the Friedmann equation are $H_0{}^2 = \frac{1}{R(t_0)^2}$, and hence $R(t_0) = H_0{}^{-1}$. Putting them together, we find $t_0 = R(t_0) = H_0{}^{-1}$, which iis $\frac{3}{2}$longer than in the matter dominated case: 1.38 Gyr, just about enough to accomodate the old stars. But this is true for an *empty* universe.

$$\left(\frac{71}{3.00\ 10^5\ 3.262\ 10^6}\right)^{-1}$$

$$1.37831 \times 10^{10}$$

Once we keep both matter and curvature, the non-flat geometry does not allow us to take $R_0 = 1$ arbitrarily. We keep $R_0 = R(t_0)$. The Friedmann equation is $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G_N \rho + \frac{k}{R^2} = \frac{8\pi}{3} G_N \rho_0 (\frac{R_0}{R})^3 + \frac{k}{R^2}$. What we can do instead is to parameterize it using $\Omega_m = \rho_0 / \rho_c$ with the critical density $\rho_c = 3 H_0{}^2 / 8\pi G_N$. Then, evaluating the equation at t_0, we find $H_0{}^2 = H_0{}^2\, \Omega_m + \frac{k}{R_0{}^2}$, and hence $R_0{}^2 = \frac{k}{H_0{}^2(1-\Omega_m)}$. From the part (a) and what we've just done, it is clear that $k = -1$

wouldn't help. We take $k = +1$ for the rest of the analysis. Then $R_0 = \left(H_0 \sqrt{1-\Omega_m}\right)^{-1}$. The Friedmann equation becomes $\left(\frac{\dot{R}}{R}\right)^2 = \frac{H_0{}^2\,\Omega_m}{R^3\,H_0{}^3 (1-\Omega_m)^{3/2}} + \frac{1}{R^2}$, and hence $d\,t = d\,R\left(1 + \frac{\Omega_m}{R\,H_0 (1-\Omega_m)^{3/2}}\right)^{-1/2}$.

Integrate[(1 + a / R)$^{-1/2}$, R]

$$\frac{\sqrt{R}\ (a+R) - a\sqrt{a+R}\ \mathrm{Log}[\sqrt{R}+\sqrt{a+R}]}{\sqrt{R}\ \sqrt{\frac{a+R}{R}}}$$

integral = Simplify[PowerExpand[%]]

$$\frac{\sqrt{R}\ (a+R) - a\sqrt{a+R}\ \mathrm{Log}[\sqrt{R}+\sqrt{a+R}]}{\sqrt{a+R}}$$

The parameter in this integral is $a = \frac{\Omega_m}{H_0 (1-\Omega_m)^{3/2}}$, while the integration region is from $R = 0$ to $R = R_0 = H_0{}^{-1} (1-\Omega_m)^{-1/2}$.

Simplify[integral /. {R → H$_0^{-1}$ (1 - Ω$_m$)$^{-1/2}$, a → Ω$_m$ H$_0^{-1}$ (1 - Ω$_m$)$^{-3/2}$}]

$$\sqrt{\frac{1}{H_0\ (1-\Omega_m)^{3/2}}}\ \sqrt{\frac{1}{H_0\ \sqrt{1-\Omega_m}}} - \frac{\mathrm{Log}\left[\sqrt{\frac{1}{H_0\ (1-\Omega_m)^{3/2}}} + \sqrt{\frac{1}{H_0\ \sqrt{1-\Omega_m}}}\ \right]\ \Omega_m}{H_0\ (1-\Omega_m)^{3/2}}$$

Simplify[integral /. {R → 0, a → Ω$_m$ H$_0^{-1}$ (1 - Ω$_m$)$^{-3/2}$}]

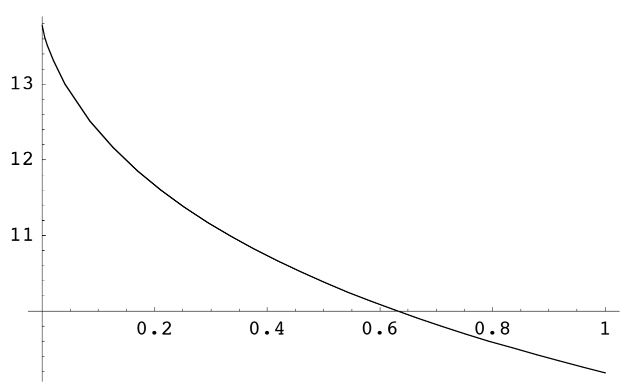
$$-\frac{\mathrm{Log}\left[\frac{\Omega_m}{H_0\ (1-\Omega_m)^{3/2}}\right]\ \Omega_m}{2\ H_0\ (1-\Omega_m)^{3/2}}$$

Simplify[PowerExpand[%% - %]]

$$-\frac{1}{4\ H_0\ (1-\Omega_m)^{3/2}}\ \left(-4\sqrt{1-\Omega_m} + \left(2\ \mathrm{Log}[H_0] + 4\ \mathrm{Log}\left[\frac{1+\sqrt{1-\Omega_m}}{\sqrt{H_0}\ (1-\Omega_m)^{3/4}}\right] + 3\ \mathrm{Log}[1-\Omega_m] - 2\ \mathrm{Log}[\Omega_m]\right)\ \Omega_m\right)$$

Clearly *Mathematica* is not smart enough to see that H_0 in the logarithms cancel with each other. I do it manually, and plot it in the unit of Gyr

Plot[$-\frac{1}{4\ H_0\ (1-\Omega_m)^{3/2}}\ \left(-4\sqrt{1-\Omega_m} + \left(4\ \mathrm{Log}\left[\frac{1+\sqrt{1-\Omega_m}}{(1-\Omega_m)^{3/4}}\right] + 3\ \mathrm{Log}[1-\Omega_m] - 2\ \mathrm{Log}[\Omega_m]\right)\ \Omega_m\right)$ /.
{H$_0$ -> 13.78^{-1}}, {Ω$_m$, 0, 1}]



- Graphics -

Knowing Ω_m is at least 0.2, it is still not in the range allowed by the age of old stars.

(c)

Instead of the curvature term, we now allow for the cosmological constant. Because of the flat geometry, we are again allowed to take $R_0 = 1$. The Friedmann equation is $\left(\frac{\dot{R}}{R}\right)^2 = H_0{}^2\,\Omega_m\,R^{-3} + H_0{}^2(1-\Omega_m)$. The equation to integrate is $d\,t = d\,R(H_0{}^2\,\Omega_m\,R^{-1} + H_0{}^2(1-\Omega_m)\,R^2)^{-1/2}$. This equation cannot be integrated as

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Integrate[H0^-1 (Ωm R^-1 + (1 - Ωm) R^2)^-1/2, R]
```

$$\frac{2\,\mathrm{Log}\left[2\left(R^{3/2}\sqrt{1-\Omega_m}+\sqrt{R^3-(-1+R^3)\,\Omega_m}\right)\right]\sqrt{R^3+\Omega_m-R^3\,\Omega_m}}{3\sqrt{R}\;H_0\sqrt{1-\Omega_m}\sqrt{\frac{R^3+\Omega_m-R^3\,\Omega_m}{R}}}$$

```
integral = Simplify[PowerExpand[%]]
```

$$\frac{2\,\mathrm{Log}\left[2\left(R^{3/2}\sqrt{1-\Omega_m}+\sqrt{R^3+\Omega_m-R^3\,\Omega_m}\right)\right]}{3\,H_0\sqrt{1-\Omega_m}}$$

```
integral /. {R → 1}
```

$$\frac{2\,\mathrm{Log}\left[2\left(1+\sqrt{1-\Omega_m}\right)\right]}{3\,H_0\sqrt{1-\Omega_m}}$$

```
integral /. {R → 0}
```

$$\frac{2\,\mathrm{Log}\left[2\sqrt{\Omega_m}\right]}{3\,H_0\sqrt{1-\Omega_m}}$$

```
Simplify[%% - %]
```

$$\frac{2\,\mathrm{Log}\left[1+\sqrt{1-\Omega_m}\right]-\mathrm{Log}[\Omega_m]}{3\,H_0\sqrt{1-\Omega_m}}$$

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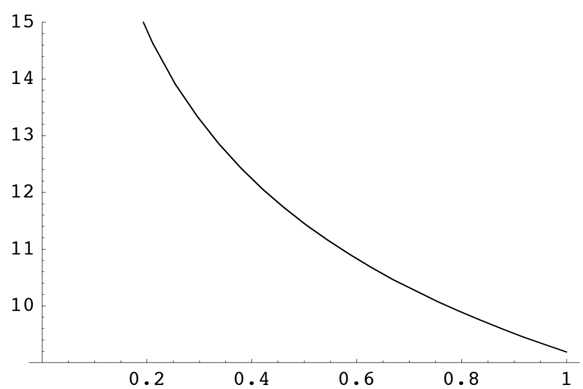
Plot[(2 Log[1 + Sqrt[1 - Ωm]] - Log[Ωm])/(3 H0 Sqrt[1 - Ωm]) /. {H0 -> 13.78^-1}, {Ωm, 0, 1}, PlotRange → {9, 15}]
```



- Graphics -

It can be clearly seen that the age of the universe can be much longer than the previous cases considered. Especially for the realistic value $\Omega_m = 0.27$, the age of the universe is 13.7 Gyr, accommodating the age of oldest stars.

$$\mathbf{N}\left[\frac{2\,\mathrm{Log}\left[1+\sqrt{1-\Omega_m}\,\right]-\mathrm{Log}[\Omega_m]}{3\,H_0\,\sqrt{1-\Omega_m}}\ /.\ \{H_0 \text{ -> } 13.78^{-1}\}\ /.\ \{\Omega_m \to 0.27\}\right]$$

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13.6792
```

2. Friedmann-Robertson-Walker metric

To work out the Ricci tensor, I use below the *Mathematica* package from http://library.wolfram.com/infocenter/MathSource/4484 that computes the Christoffel symbol, Riemann curvature, and Ricci tensor, etc, from the metric. If you download the package, read it in from the location.

```
<< ~/Documents/229C/GREAT.m

GREAT functions are: IMetric, Christoffel,
  Riemann, Ricci, SCurvature, EinsteinTensor, SqRicci, SqRiemann.

Enter 'helpGREAT' for this list of functions
```

First define the coordinate n-vector:

```
x = {t, r, θ, ϕ}

{t, r, θ, ϕ}
```

and then specify the metric as a square $n \times n$ matrix:

$$\left(\text{met} = \left\{\{1, 0, 0, 0\}, \left\{0, \frac{R[t]^2}{1 - k\,r^2}, 0, 0\right\}, \{0, 0, R[t]^2\,r^2, 0\}, \{0, 0, 0, R[t]^2\,r^2\,\mathrm{Sin}[\theta]^2\}\right\}\right) \text{ // MatrixForm}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{R[t]^2}{1-k\,r^2} & 0 & 0 \\ 0 & 0 & r^2\,R[t]^2 & 0 \\ 0 & 0 & 0 & r^2\,R[t]^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

```
Ricci[met, x]
```

$$\left\{\left\{-\frac{3\,R''[t]}{R[t]}, 0, 0, 0\right\}, \left\{0, \frac{-2\,k + 2\,R'[t]^2 + R[t]\,R''[t]}{-1 + k\,r^2}, 0, 0\right\}, \{0, 0, r^2\,(2\,k - 2\,R'[t]^2 - R[t]\,R''[t]), 0\}, \{0, 0, 0, r^2\,\mathrm{Sin}[\theta]^2\,(2\,k - 2\,R'[t]^2 - R[t]\,R''[t])\}\right\}$$

```
Simplify[Inverse[met].%]
```

$$\left\{\left\{-\frac{3\,R''[t]}{R[t]}, 0, 0, 0\right\}, \left\{0, \frac{2\,k - 2\,R'[t]^2 - R[t]\,R''[t]}{R[t]^2}, 0, 0\right\}, \left\{0, 0, \frac{2\,k - 2\,R'[t]^2 - R[t]\,R''[t]}{R[t]^2}, 0\right\}, \left\{0, 0, 0, \frac{2\,k - 2\,R'[t]^2 - R[t]\,R''[t]}{R[t]^2}\right\}\right\}$$

I'm afraid I had a typo in the problem set: the term with k has the opposite sign! Sorry about that.