HW #4

1. isothermal halo

The isothermal model of halo assumes the distribution
$\rho(\vec{x}, \vec{v}) = N \exp\left(\frac{1}{\sigma^2}\left(\Psi - \frac{1}{2}\vec{v}^2\right)\right)$.
This ansatz is called *isothermal* because of its resemblance to the Boltzmann distribution
$\rho(\vec{x}, \vec{v}) = N \exp\left(\frac{1}{m\sigma^2}\left(m\Psi - \frac{m}{2}\vec{v}^2\right)\right) = N e^{-E/kT}$
with the identifications $kT = m\sigma^2$ and the single-particle energy $E = \frac{m}{2}\vec{v}^2 - m\Psi$. (The griavitational potential energy is $V = m\Psi$.) But this is not just a resemblance. Any phase distribution function given in terms of the Hamiltonian is automatically a static solution to the Boltzmann equation, and furthermore an exponential form (Boltzmann distribution in thermal equilibrium) is known to be a particularly robust solution under small perturbations. Therefore, it is a good guess for a stable configuration of the halo.

The spatial density is given by the integration upon the velocities,
$\rho(r) = \int d^3 v\, N \exp\left(\frac{1}{\sigma^2}\left(\Psi - \frac{1}{2}\vec{v}^2\right)\right) = (2\pi\sigma^2)^{3/2} N e^{\Psi/\sigma^2}$.
With the boundary condition $\rho(0) = \rho_0$, $(2\pi\sigma^2)^{3/2} N e^{\Psi(0)/\sigma^2} = \rho_0$, and hence
$\rho(r) = \rho_0\, e^{(\Psi(r)-\Psi(0))/\sigma^2}$. Namely,
$\Psi(r) = \Psi(0) + \sigma^2 \log\frac{\rho(r)}{\rho_0}$.
The Poisson equation is then
$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\sigma^2 \log\frac{\rho(r)}{\rho_0}\right) = -4\pi G\rho(r)$.
Now using the variables $\tilde{\rho} = \rho/\rho_0$, $r = r_0\tilde{r}$, $r_0 = \sqrt{9\sigma^2/4\pi G\rho_0}$,
$\frac{\sigma^2}{r_0{}^2}\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}(\tilde{r}^2\frac{d}{d\tilde{r}}\log\tilde{\rho}) = -4\pi G\rho_0\tilde{\rho}$
or
$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}(\tilde{r}^2\frac{d}{d\tilde{r}}\log\tilde{\rho}) = -r_0{}^2\frac{4\pi G\rho_0}{\sigma^2}\tilde{\rho} = -9\tilde{\rho}$
Writing out the derivatives,
$2\frac{1}{\tilde{r}}\frac{\tilde{\rho}'}{\tilde{\rho}} + \frac{\tilde{\rho}''}{\tilde{\rho}} - \left(\frac{\tilde{\rho}'}{\tilde{\rho}}\right)^2 = -9\tilde{\rho}$
subject to the boundary condition $\tilde{\rho}(0) = 1$. We also impose $\tilde{\rho}'(0) = 0$to avoid a $\frac{1}{\tilde{r}}$singularity in the above equation at the origin. Then we find $\tilde{\rho}''(0) = -3$.

```
solution = NDSolve[{2 If[x == 0, -3, y'[x] / x] / y[x] + (y''[x] / y[x] - (y'[x] / y[x])^2) == -9 y[x],
    y[0] == 1, y'[0] == 0}, y, {x, 0, 100}]

{{y → InterpolatingFunction[{{0., 100.}}, <>]}}

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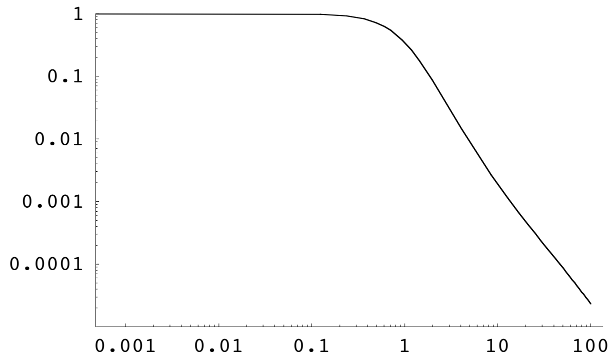
```
LogLogPlot[Evaluate[y[x] /. solution], {x, 0, 100}, PlotRange → {0.00001, 1}]
```



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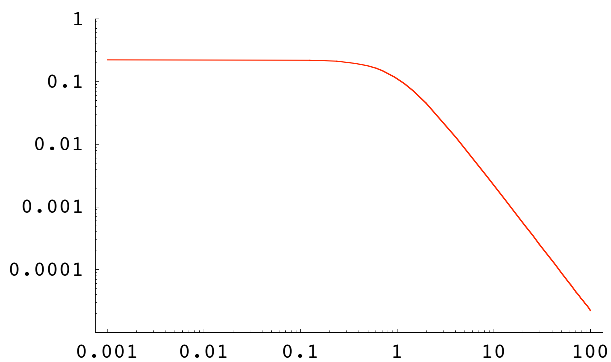
```
LogLogPlot[2/9 1/(1 + x^2), {x, 0.001, 100},
 PlotRange → {0.00001, 1}, PlotStyle → RGBColor[1, 0, 0]]
```



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Show[%, %%]
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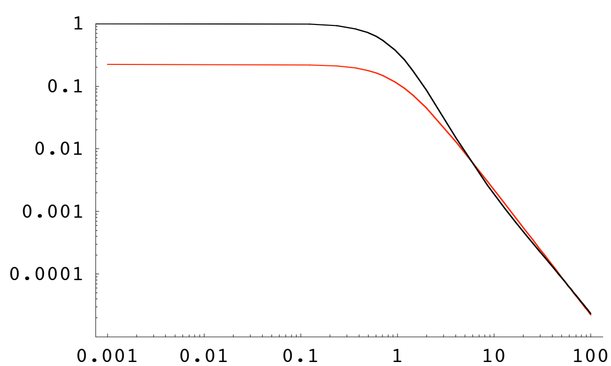

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Indeed, the asymptotic behavior is given approximately by $\tilde{\rho} = \frac{2}{9} \frac{1}{1+\tilde{r}^2}$.

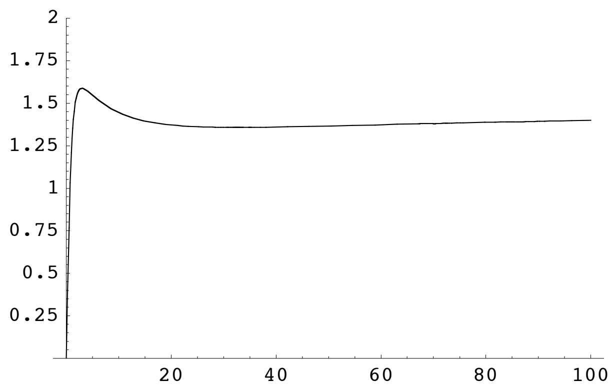
This behavior can also be understood analytically. Going back to the differential equation

$2\,\tilde{r}\,\frac{\tilde{\rho}'}{\tilde{\rho}} + \tilde{r}^2\,\frac{\tilde{\rho}''}{\tilde{\rho}} - \tilde{r}^2\left(\frac{\tilde{\rho}'}{\tilde{\rho}}\right)^2 = -9\,\tilde{r}^2\,\tilde{\rho}$
and assume the asymptotic behavior $\tilde{\rho} \sim A\,\tilde{r}^{-n}$. Keeping only the leading terms in $1/\tilde{r}$,
$2\,(-n) + (-n)\,(-n-1) - n^2 = -9\,A\,\tilde{r}^{2-n}$
If $n < 2$, there is only a trivial solution $A = 0$. If $n > 2$, $-2\,n + n(n+1) - n^2 = 0$ and hence $n = 0$, a contradiction. Therefore, $n = 2$, and then $A = \frac{2}{9}$.

The rotation speed of stars embedded inside the halo is determined by the usual balance between the gravitational force and the centrifugal force,
$m\,\frac{v^2}{r} = \frac{dV}{dr} = -m\,\frac{d\Psi}{dr} = -m\,\sigma^2\,\frac{d\rho/dr}{\rho}.$
Therefore,
$v^2 = -\sigma^2\,\frac{r}{\rho}\,\frac{d\rho}{dr} = -\sigma^2\,\tilde{r}\,\frac{\tilde{\rho}'}{\tilde{\rho}}.$
Note that the asymptotic rotation speed is found using the asymptotic solution obtained above,
${v_\infty}^2 = -\sigma^2\,\tilde{r}(-2)\,\tilde{r}^{-1} = 2\,\sigma^2$, and hence $v_\infty = \sqrt{2}\,\sigma$.

The fact that the rotation speed approaches a constant instead of falling as $r^{-1/2}$ is the most surprising feature of the data ("flat rotation curve"), which is reproduced in this model.

```
Plot[Sqrt[Evaluate[-x y'[x]/y[x] /. solution]], {x, 0.001, 100}, PlotRange → {0, 2}]
```



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2. Gravitational Microlensing

It is presented in a separate PDF because I couldn't insert figures into *Mathematica* notebook.