# Gravitational Microlensing

## 1 Amplification

Once the deflection angle is known (see the optional problem), it is easy to work out the amplification using simple geometric optics. Throughout the discussion, we keep only the first order in very small angles. Just by looking at the geometry in Fig. 1, the deflection angle is

$$\Delta\theta = \theta_1 + \theta_2 = \frac{r - r_0}{d_1} + \frac{r - r_0}{d_2} = \frac{4G_N m}{r}. \tag{1}$$

Here, $r_0$ is the impact parameter. When $r_0 = 0$ (exactly along the line of sight), the solution is simple:

$$r(r_0 = 0) = R_0 \equiv \sqrt{4G_N m \frac{d_1 d_2}{d_1 + d_2}}. \tag{2}$$

This is what is called the Einstein radius, $R_0$ in Packzynski's notation (B. Paczynski, *Ap. J.* **304**, 1–5 (1986), `http://adsabs.harvard.edu/cgi-bin/nph-bib_query?bibcode=1986ApJ...304....1P`). For general $r_0$, Eq. (1) can be rewritten as

$$r(r - r_0) - R_0^2 = 0, \tag{3}$$

which is Eq. (1) in the Packzynski's paper. It has two solutions

$$r_\pm(r_0) = \frac{1}{2}\left(r_0 \pm \sqrt{r_0^2 + 4R_0^2}\right). \tag{4}$$

The solution with the positive sign is what is depicted in Fig. 1, while the solution with the negative sign makes the light ray go *below* the lens.

To figure out the amplification due to the gravitational lensing, we consider the finite aperture of the telescope (*i.e.*, the size of the mirror). We assume an infinitesimal circular aperture. From the point of view of the star, the finite aperture is an image on the deflection plane, namely the plane perpendicular to the straight line from the star to the telescope where the lens is. The vertical aperture changes the impact parameter $r_0$ to a range $r_0 \pm \delta$ (size of the mirror is $\delta \times (d_1 + d_2)/d_2$). Correspondingly, the image of

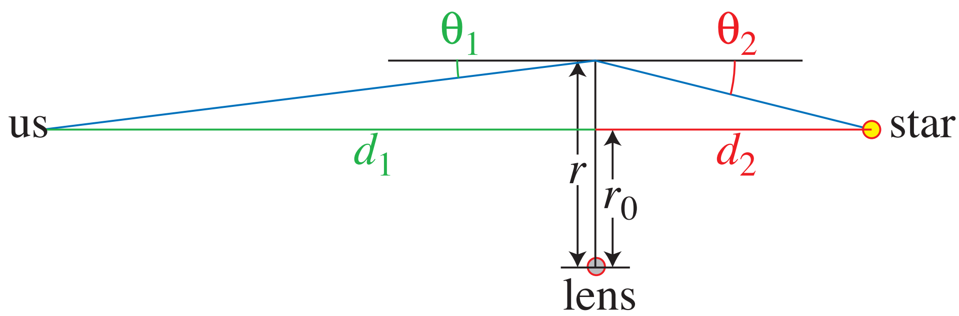


Figure 1: The deflection of light due to a massive body close to the line of sight towards a star.

the telescope is at $r_\pm(r_0 \pm \delta) = r_\pm(r_0) \pm \delta \frac{dr_\pm}{dr_0}$.[1] Using the solution Eq. (4), we find that the vertical aperture always appears squashed (see Fig. 2),

$$\delta \times \left| \frac{dr}{dr_0} \right| = \delta \times \left| \frac{1}{2} \left( 1 \pm \frac{r_0}{\sqrt{r_0^2 + 4R_0^2}} \right) \right| = \delta \times \frac{\sqrt{r_0^2 + 4R_0^2} \pm r_0}{2\sqrt{r_0^2 + 4R_0^2}} < \delta. \tag{5}$$

On the other hand, the horizontal aperture is scaled as

$$\delta \times \frac{r}{r_0}. \tag{6}$$

Because the amount of light that goes into the mirror is proportional to the elliptical aperture from the point of view of the star that emits light isotropically, the magnification is given by

$$A_\pm = \frac{r}{r_0} \left| \frac{dr}{dr_0} \right| = \frac{(\sqrt{r_0^2 + 4R_0^2} \pm r_0)^2}{4r_0\sqrt{r_0^2 + 4R_0^2}} = \frac{2r_0^2 + 4R_0^2 \pm 2r_0\sqrt{r_0^2 + 4R_0^2}}{4r_0\sqrt{r_0^2 + 4R_0^2}}. \tag{7}$$

The total magnificiation sums two images,

$$A = A_+ + A_- = \frac{r_0^2 + 2R_0^2}{r_0\sqrt{r_0^2 + 4R_0^2}} = \frac{u^2 + 2}{u\sqrt{u^2 + 4}} \tag{8}$$

[1]Note that this Taylor expansion is valid only when $\delta \ll r_0$. For $\delta \sim r_0$, we have to work it out more precisely; see next section.

with $u = r_0/R_0$.[2] Basically, there is a significant amplification of the brightness of the star when the lens passes through the line of sight within the Einstein radius.

We estimate the frequency and duration of gravitational microlensing due to MACHOs in the galactic halo. The Large Magellanic Cloud is about 50kpc away from us, while we are about 8.5kpc away from the galactic center. The flat rotation curve for the Milky Way galaxy is about 220 km/sec (see Fig. 6 in the Raffelt's review). The Einstein radius for a MACHO is calculated from Eq. (2),

$$R_0 = \sqrt{\frac{4G_N m}{c^2}\frac{d_1 d_2}{d_1+d_2}} = 1.2410^{12}\ \mathrm{m}\left(\frac{m}{M_\odot}\right)^{1/2}\left(\frac{\sqrt{d_1 d_2}}{25\mathrm{kpc}}\right). \tag{9}$$

To support the rotation speed of $v_\infty = 220$ km/sec in the isothermal model of halo, we need the velocity dispersion $\sigma = v_\infty/\sqrt{2}$. The average velocity transverse to the line of sight is

$$\langle v_x^2 + v_y^2\rangle = 2\sigma^2 = v_\infty^2. \tag{10}$$

The time it takes a MACHO to traverse the Einstein radius is

$$\frac{R_0}{v_\infty} = 5.6\times 10^6\,\mathrm{sec}\left(\frac{m}{M_\odot}\right)^{1/2}\left(\frac{\sqrt{d_1 d_2}}{25\mathrm{kpc}}\right), \tag{11}$$

about two months for $m = M_\odot$ and $d_1 = d_2 = 25$ kpc. A microlensing event of duration shorter than a year can be in principle be seen.[3]

The remaining question is the frequency of such microlensing events. It is the probability of a randomly moving MACHO coming within the Einstein radius of a star in the LMC. We will make a crude estimate. The flat rotation curves requires $\frac{G_N M(r)}{r^2} = \frac{v_\infty^2}{r}$ and hence the halo density $\rho(r) = \frac{v_\infty^2}{4\pi G_N r^2}$. The number density of MACHOs, assuming they dominate the halo, is then $n(r) = \frac{v_\infty^2}{4\pi G_N m r^2}$. Instead of dealing with the Boltzmann (Gaussian) distribution in velocities, we simplify the problem by assuming that $\vec{v}_\perp^2 = v_x^2 + v_y^2 = \sigma^2$.

[2] The singular behavior for $r_0 \to 0$ is due to the invalid Taylor expansion in $\delta$. This is practically not a concern because it is highly unlikely that a MACHO passes through with $r_0 \lesssim \delta_0$. Note that the true image is actually not quite elliptic but distorted in this case.

[3] MACHO collaboration did even more patient scanning to look for microlensing events longer than a year, *Astrop. J. Lett.* **550**, L169 (2001), `http://www.journals.uchicago.edu/ApJ/journal/issues/ApJL/v550n2/005978/brief/005978.abstract.html` (astro-ph/0011506).

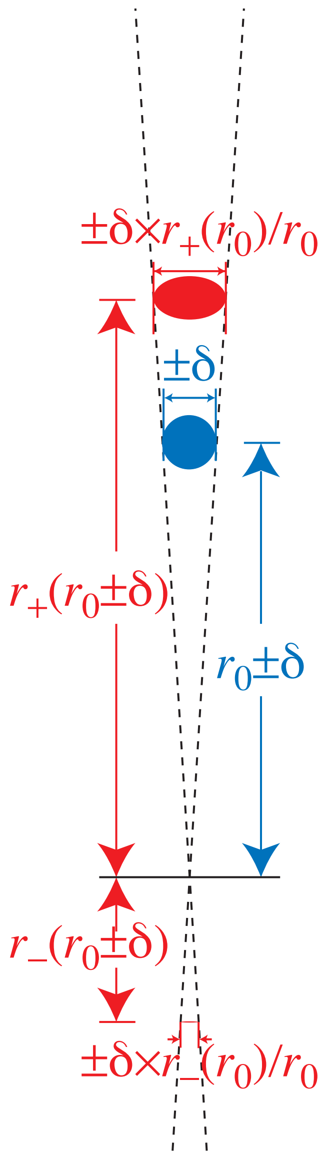


Figure 2: The way the mirror of the telescope appears on the deflection plane from the point of view of the star. For the purpose of illustration, we took $R_0 = 2$, $r_0 = 3$.

From the transverse distance $r_\perp = \sqrt{x^2+y^2}$, only the fraction $R_0/r_\perp$ heads the right direction for the distance $\sigma\Delta t$. Therefore the fraction of MACHOs that pass through the Einstein radius is

$$\int_0^{\sigma\Delta t} 2\pi r_\perp dr_\perp \frac{R_0}{r_\perp} = 2\pi R_0 \sigma \Delta t. \tag{12}$$

We then integrate it over the depth with the number density. The distance from the solar system to the LMA is not the same as the distance from the galactic center because of the relative angle $\alpha = 82°$. The solar system is away from the galactic center by $r_\odot = 8.5$ kpc. Along the line of sight to the LMA with depth $R$, the distance from the galactic center is given by $r^2 = R^2 + r_\odot^2 - 2Rr_\odot \cos\alpha$ with $\alpha = 82°$. Therefore the halo density along the line of sight is

$$n(r) = \frac{v_\infty^2}{4\pi G_N m(R^2 + r_\odot^2 - 2Rr_\odot\cos\alpha)} \tag{13}$$

The number of MACHOs passing through the line of sight towards a star in the LMA within the Einstein radius is

$$\int_0^{R_{LMC}} dR n(r) 2\pi R_0 \sigma = \int_0^{R_{LMC}} dR \frac{v_\infty^2}{4\pi G_N m(R^2 + r_\odot^2 - 2Rr_\odot\cos\alpha)} 2\pi R_0 \sigma \Delta t \tag{14}$$

Pakzynski evaluates the optical depth, but I'd rather estimate a quantity that is directly relevant to the experiment, namely the rate of the microlensing events. Just by taking $\Delta t$ away,

$$\begin{aligned} \text{rate} &= \int_0^{R_{LMC}} dR \frac{v_\infty^2}{4\pi G_N m(R^2 + r_\odot^2 - 2Rr_\odot\cos\alpha)} 2\pi R_0 \sigma \\ &= \int_0^{R_{LMC}} dR \frac{v_\infty^2}{R^2 + r_\odot^2 - 2Rr_\odot\cos\alpha} \sqrt{\frac{R(R_{LMC}-R)}{G_N m R_{LMC}}} \frac{\sigma}{c} \\ &= \frac{v_\infty^2 \sigma}{c\sqrt{G_N m R_{LMC}}} \int_0^{R_{LMC}} \frac{\sqrt{R(R_{LMC}-R)}\, dR}{R^2 + r_\odot^2 - 2Rr_\odot\cos\alpha}. \end{aligned} \tag{15}$$

The integral can be evaluated numerically. For $R_{LMC} = 50$ kpc, $r_\odot = 8.5$ kpc, $\alpha = 82°$, Mathematica gives 3.05. Then with $\sigma = v_\infty/\sqrt{2}$, $v_\infty = 220$ km/sec, we find

$$\text{rate} = 1.69 \times 10^{-13}\,\text{sec}^{-1} \left(\frac{M_\odot}{m}\right)^{1/2} = 5.34 \times 10^{-6}\text{year}^{-1} \left(\frac{M_\odot}{m}\right)^{1/2}. \tag{16}$$

Therefore, if we can monitor about a million stars, we may see 5 microlensing events for a solar mass MACHO, even more for ligher ones.

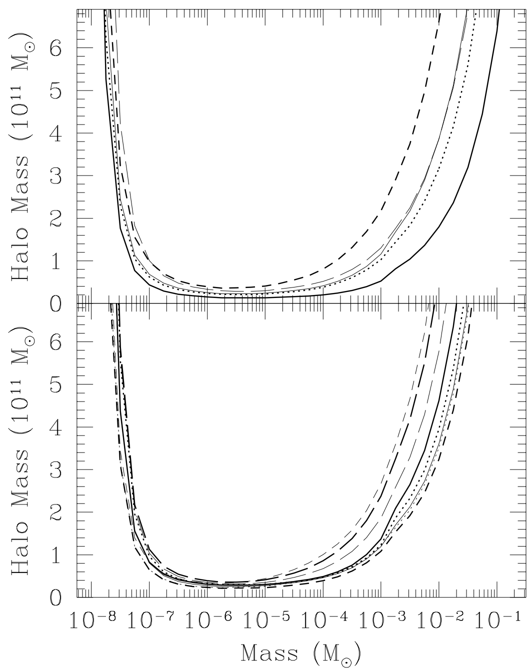


Figure 3: The limit on the MACHO fraction of the halo, combining data from MACHO and EROS collaborations, *Astrop. J. Lett.*, **499**, L9 (1998) `http://www.journals.uchicago.edu/ApJ/journal/issues/ApJL/v499n1/985171/985171.web.pdf`

# 2 Strong Lensing

Even though it is not a part of this problem, it is fun to see what happens when $r_0 \lesssim \delta$. This can be studied easily with a slightly tilted coordinates in Fig. 4.

Using this coordinate system, we can draw a circle on the plane $(x, y) = (x_0, y_0) + \rho(\cos\phi, \sin\phi)$, and the corresponding image on the deflector plane is $(\tilde{x}, \tilde{y}) = (\tilde{x}_0, \tilde{y}_0) + \tilde{\rho}(\cos\phi, \sin\phi) = \frac{d_2}{d_1+d_2}(x, y)$. The impact parameter is then $r_0 = \sqrt{\tilde{x}^2 + \tilde{y}^2}$ which allows us to calculate $r_\pm(r_0)$ using Eq. (4) for each $\phi$. Obviously $\phi$ is the same for the undistorted and distorted images.

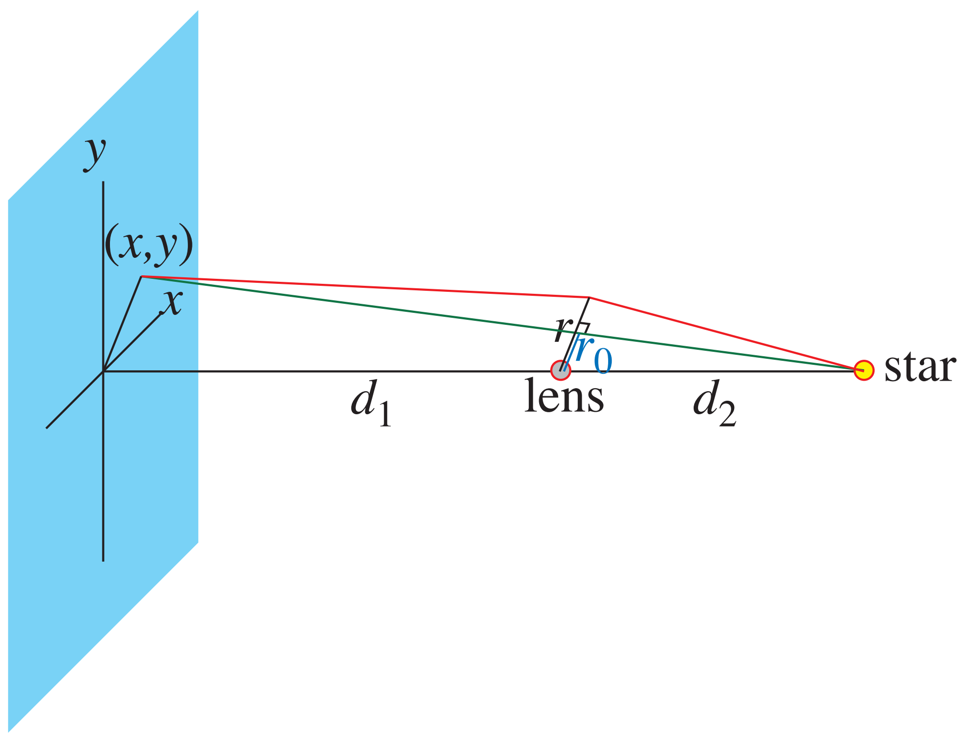


Figure 4: A slightly different coordinate system to work out the distortion of images.

Fig. 5 shows a spectacular example with $(x_0, y_0) = (1, 0)$, $\frac{d_2}{d_1+d_2} = \frac{1}{3}$, $\rho = 0.8$. Because $\rho \sim r_0$, the Taylor expansion does not work, and the image is far from an ellipse.

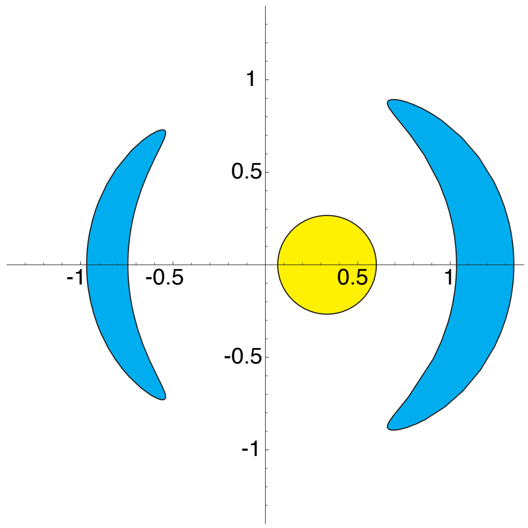


Figure 5: A highly distorted image due to the gravitational lensing. Yellow circle is the undistorted image, while the two blue regions are the images distorted by the gravitational lensing.

This kind of situation is not expected to occur for something as small as the mirror of a telescope, but may for something as big as a galaxy. When an image of a galaxy is distorted by a concentration of mass in the foreground, such as a cluster of galaxies, people have seen spectacular "strong lensing" effects.

# 3 Derivation of the Deflection Angle

For the introduction to Hamilton–Jacobi equations, see `http://hitoshi.berkeley.edu/221A/classical2.pdf`.

Using the Schwarzschild metric ($c = 1$)

$$ds^2 = \frac{r - r_S}{r} dt^2 - \frac{r}{r - r_S} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \tag{17}$$

Figure 6: A Hubble Space Telescope image of a gravitational lens formed by the warping of images of objects behind a massive concentration of dark matter. Warped images of the same blue background galaxy are seen in multiple places. Taken from `http://www.bell-labs.com/org/physicalsciences/projects/darkmatter/darkmatter1.html`.

where $r_S = 2G_N m$ is the Schwarzschild radius. The Hamilton–Jacobi equation for light in this metric is

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = \frac{r}{r-r_S}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{r-r_S}{r}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 = 0. \tag{18}$$

We separate the variables as

$$S(t,r,\theta,\phi) = S_1(t) + S_2(r) + S_3(\theta) + S_4(\phi) \tag{19}$$

where

$$\frac{r}{r-r_S}\left(\frac{dS_1}{dt}(t)\right)^2 - \frac{r-r_S}{r}\left(\frac{dS_2}{dr}(r)\right)^2 - \frac{1}{r^2}\left(\frac{dS_3}{d\theta}(\theta)\right)^2 - \frac{1}{r^2\sin^2\theta}\left(\frac{dS_4}{d\phi}(\phi)\right)^2 = 0. \tag{20}$$

Because the equation does not contain $t$ or $\phi$ explicitly, their functions must be constants,

$$\frac{dS_1}{dt} = -E, \tag{21}$$

$$\frac{dS_4}{d\phi} = L_z. \tag{22}$$

We can solve them immediately as

$$S_1(t) = -Et, \tag{23}$$
$$S_4(\phi) = L_z\phi. \tag{24}$$

Then Eq. (18) becomes

$$\frac{r}{r-r_S}E^2 - \frac{r-r_S}{r}\left(\frac{dS_2}{dr}(r)\right)^2 - \frac{1}{r^2}\left(\frac{dS_3}{d\theta}(\theta)\right)^2 - \frac{1}{r^2\sin^2\theta}L_z^2 = 0. \tag{25}$$

The $\theta$ dependence is only in the last two terms and hence

$$\left(\frac{dS_3}{d\theta}(\theta)\right)^2 + \frac{1}{\sin^2\theta}L_z^2 = L^2 \tag{26}$$

is a constant which can be integrated explicitly if needed. Without a loss of generality, we can choose the coordinate system such that the orbit is on the $x$-$y$ plane, and hence $L_z = 0$. In this case, $S_4(\phi) = 0$ and $S_3 = L\theta$. Finally, the equation reduces to

$$\frac{r}{r-r_S}E^2 - \frac{r-r_S}{r}\left(\frac{dS_2}{dr}(r)\right)^2 - \frac{L^2}{r^2} = 0. \tag{27}$$

Therefore,

$$S_2(r) = \int\sqrt{\frac{r^2}{(r-r_S)^2}E^2 - \frac{L^2}{r(r-r_S)}}\,dr. \tag{28}$$

Since $S(t,r,\theta,\phi) = S_2(r) - Et + L\theta$, $S_2$ can be regarded as Legendre transform $S_2(r,E,L)$ of the action, and hence the inverse Legendre transform gives

$$\frac{\partial S_2(r,E,L)}{\partial L} = -\theta. \tag{29}$$

Using the expression Eq. (28), we find

$$\theta(r) = \int_{r_c}^{r}\frac{Ldr}{\sqrt{E^2r^4 - L^2r(r-r_S)}}\ . \tag{30}$$

The closest approach is where the argument of the square root vanishes,

$$E^2r_c^4 - L^2r_c(r_c - r_S) = 0. \tag{31}$$

It is useful to verify that the $m = 0$ ($r_S = 0$) limit makes sense. The closest approach is $E^2 r_c^4 - L^2 r_c^2 = 0$ and hence $r_c = L/E$, which is the impact parameter. The orbit Eq. (30) is

$$\theta(r) = \int_{r_c}^{r} \frac{L dr}{\sqrt{E^2 r^4 - L^2 r^2}} = \int_{r_c}^{r} \frac{r_c dr}{r\sqrt{r^2 - r_c^2}}. \tag{32}$$

Change the variable to $r = r_c \cosh\eta$, and we find

$$\theta(r) = \int_0^{\eta} \frac{r_c^2 \sin\eta d\eta}{r_c \cosh\eta r_c \sinh\eta} = \int_0^{\eta} \frac{d\eta}{\cosh\eta} = 2 \arctan\tanh\frac{\eta}{2}. \tag{33}$$

Hence $\tan\frac{\theta}{2} = \tanh\frac{\eta}{2}$, and

$$\cos\theta = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{1 - \tanh^2\eta/2}{1 + \tanh^2\eta/2} = \frac{1}{\cosh\eta} = \frac{r_c}{r}. \tag{34}$$

Therefore $r_c = r\cos\theta$ which is nothing but a straight line.

To find the deflection angle, we only need to calculate the asymptotic angle $\theta(r = \infty)$. Going back to Eq. (30), we need to calculate

$$\theta(\infty) = \int_{r_c}^{\infty} \frac{L dr}{\sqrt{E^2 r^4 - L^2 r(r - r_S)}} . \tag{35}$$

We would like to expand it up to the linear order in $r_S \ll r_c$. If you naively expand the integrand in $r_S$, the argument of the square root in the resulting expression can be negative for $r = r_c < L/E$. To avoid this problem, we change the variable to $r = r_c/x$:

$$\theta(\infty) = \int_0^1 \frac{L r_c dx}{\sqrt{E^2 r_c^4 - L^2 r_c (r_c - r_S x) x^2}} . \tag{36}$$

Using Eq. (31), we write $E^2 r_c^4$ and obtain

$$\theta(\infty) = \int_0^1 \frac{r_c dx}{\sqrt{r_c^2(1 - x^2) - r_c r_S (1 - x^3)}} . \tag{37}$$

Expanding it to the linear order in $r_S/r_c$, we find

$$\theta(\infty) = \int_0^1 \left( \frac{1}{\sqrt{1 - x^2}} + \frac{(1 + x + x^2) r_S}{2(1 + x)\sqrt{1 - x^2}\, r_c} + O(r_S)^2 \right) dx = \frac{\pi}{2} + \frac{r_S}{r_c}. \tag{38}$$

The deflection angle is $\Delta\theta = \pi - 2\theta(\infty) = 2\frac{r_S}{r_c} = 4G_N m/r_c$. It is easy to recover $c = 1$ by looking at the dimensions, and we find $\Delta\theta = 4G_N m/c^2 r_c$.

It is also useful to know the closest approach $r_c$ to the first order in $m$. We expand $r_c$ as $r_c = \frac{L}{E} + \Delta$. Then Eq. (31) gives

$$4\frac{L^3}{E}\Delta - 2\frac{L^3}{E}\Delta + \frac{E^3}{L}r_S + O(r_S)^2 = 0, \tag{39}$$

and hence

$$r_c = \frac{L}{E} - \frac{r_S}{2} + O(r_S)^2. \tag{40}$$