HW #2

(1) τ-decay

The electron or muon four-momentum in the τ rest frame is

$$p_l^\mu = y\,\frac{1}{2}\,m_\tau(1,\ \sin\theta,\ 0,\ \cos\theta),$$

by suitably choosing the orientation of the (1, 2) plane. Here I've dropped "hat" on θ, and renamed x-hat as y just to save time to type. By boosting it to the lab frame (with $\gamma = m_Z/(2\,m_\tau)$ and taking the approximation $\beta = 1$), it is

$$p_l^\mu = y\,\frac{1}{2}\,m_\tau(\gamma\,(1+\cos\theta),\ \sin\theta,\ 0,\ \gamma\,(1+\cos\theta)) = y\,\frac{m_Z}{4}\left((1+\cos\theta),\ \frac{2\,m_\tau}{m_Z}\sin\theta,\ 0,\ (1+\cos\theta)\right)$$

Therefore,

$$x = \frac{2\,p_l^0}{m_Z} = \frac{1}{2}\,y\,(1+\cos\theta).$$

Now it is compute the distribution in x by integrating over y and $\cos\theta$ with the delta function $\delta\left(x - \frac{1}{2}\,y(1+\cos\theta)\right)$.

$$\frac{d\Gamma}{d\,x} = \int_0^1 d\,y \int_{-1}^1 d\cos\theta\,\delta\left(x - \frac{1}{2}\,y(1+\cos\theta)\right)\frac{G_F^{\,2}\,m_\tau^{\,5}}{192\,\pi^3}[3 - 2\,y \pm \cos\theta(2\,y-1)]\,y^2$$

$$= \int_x^1 d\,y\,\frac{2}{y}\,\frac{G_F^{\,2}\,m_\tau^{\,5}}{192\,\pi^3}\left[3 - 2\,y \pm \left(\frac{2\,x}{y} - 1\right)(2\,y-1)\right]y^2$$

$$= \frac{G_F^{\,2}\,m_\tau^{\,5}}{192\,\pi^3}\int_x^1 d\,y\,2\left[3\,y - 2\,y^2 \pm (2\,x - y)\,(2\,y-1)\right]$$

$$= \frac{G_F^{\,2}\,m_\tau^{\,5}}{96\,\pi^3}\left[\frac{2}{3}\left(1-x^3\right),\ 1 - 3\,x^2 + 2\,x^3\right]$$

where the last expression is for + and − sign, respectively.

```
Integrate[3 y - 2 y^2 + (2 x - y) (2 y - 1), {y, x, 1}]
```

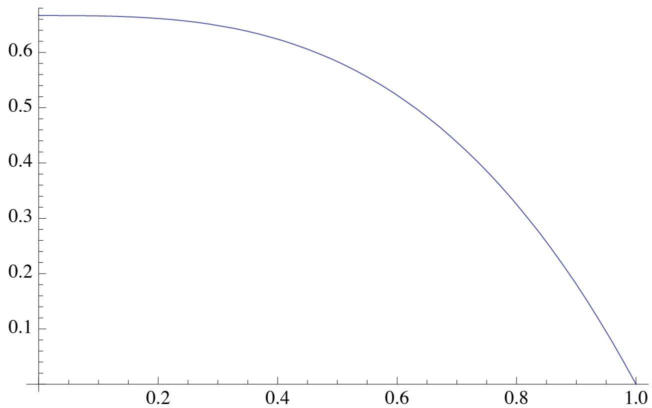
$$\frac{2}{3} - \frac{2\,x^3}{3}$$

```
Integrate[3 y - 2 y^2 - (2 x - y) (2 y - 1), {y, x, 1}]
```

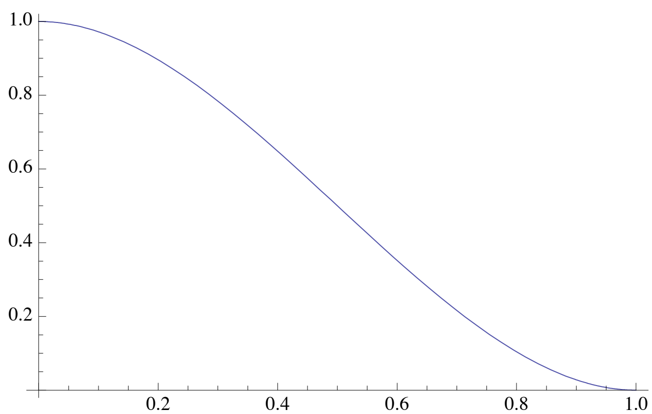
$$1 - 3\,x^2 + 2\,x^3$$

We plot them:

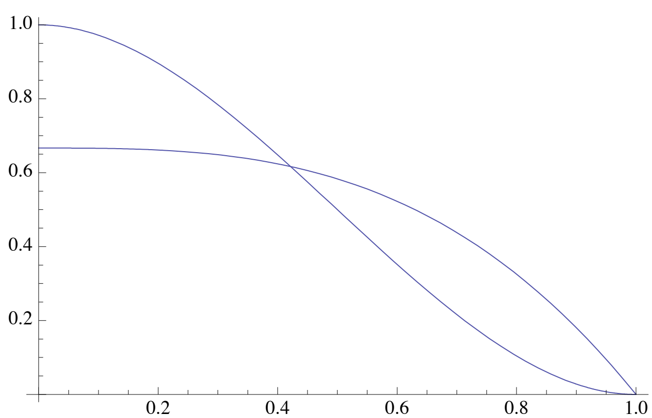
```
Plot[2/3 (1 - x^3), {x, 0, 1}]
```




```
Plot[1 - 3 x^2 + 2 x^3, {x, 0, 1}]
```




```
Show[%, %%]
```

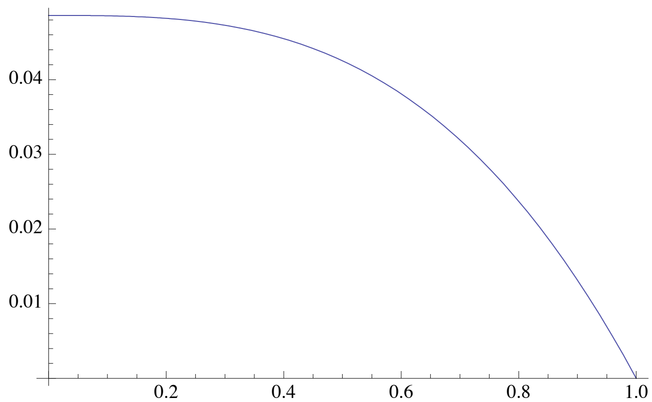


The spectra resemble closely what we see in Fig. 13(a), (b) of the ALEPH paper, except for low x which is clealy affected by acceptance bias. In order to make them appear even closer, we can plot them with the correct relative fraction $f_R = \left(\sin^2\theta_W\right)^2$, $f_L = \left(\frac{-1}{2} + \sin^2\theta_W\right)^2$.

In[12]:= `Plot[(-0.5 + sW^2)^2 2/3 (1 - x^3) /. {sW → Sqrt[0.23]}, {x, 0, 1}]`

$$\text{Plot}\left[\left(-0.5+s_W{}^2\right)^2 \frac{2}{3} \left(1-x^3\right) \text{ /. } \left\{s_W \to \sqrt{0.23}\right\}, \{x, 0, 1\}\right]$$

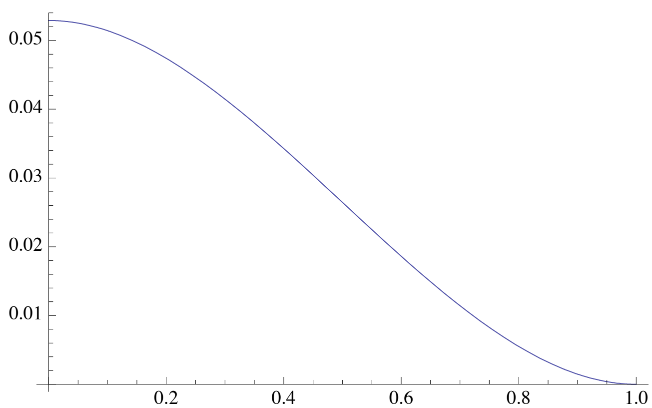
Out[12]=



In[13]:=

$$\text{Plot}\left[\left(s_W{}^2\right)^2 \left(1-3\,x^2+2\,x^3\right) \text{ /. } \left\{s_W \to \sqrt{0.23}\right\}, \{x, 0, 1\}\right]$$

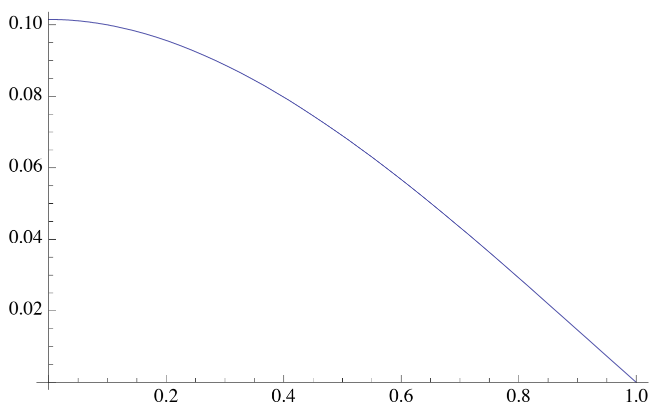
Out[13]=



In[14]:=

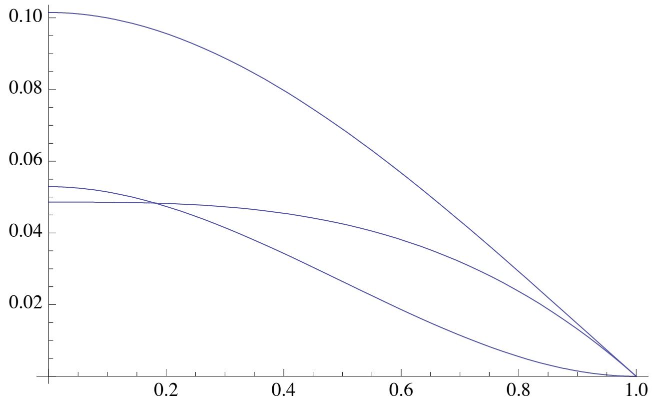
$$\text{Plot}\left[\left(-0.5+s_W{}^2\right)^2 \frac{2}{3} \left(1-x^3\right) + \left(s_W{}^2\right)^2 \left(1-3\,x^2+2\,x^3\right) \text{ /. } \left\{s_W \to \sqrt{0.23}\right\}, \{x, 0, 1\}\right]$$

Out[14]=



In[15]:= `Show[%, %%, %%%]`

Out[15]=



This plot looks basically the same as that in the paper except for very low x. In the actual experiment, they fit the lepton spectrum keeping s_W free based on the predicted distributions for both left- and right-handed tau.

(2) impact parameter distribution

Suppose the long-lived particle traveled over the distance l and decayed into a charged track along the polar angle θ relative to the original direction. There may of course be several of them, but we will use only one. We assume that the track is massless. We define the coordinate system such that the original direction is the z-axis.

In the rest frame of the decaying particle, the charged track has the four-momentum $\hat{p} = E(1, \sin\theta, 0, \cos\theta)$. (I omit the hat on θ just for saving time with typing.) Going go the lab frame, we use the boost factor $\beta = v/c$ and $\gamma = 1 \Big/ \sqrt{1-\beta^2}$. The four-momentum of the charged track then is $p = E(\gamma + \gamma\,\beta\cos\theta,\ \sin\theta,\ 0,\ \gamma\cos\theta + \gamma\,\beta)$. Therefore, the track is pointing to the direction $\vec{n} = (\sin\theta,\ 0,\ \gamma\cos\theta + \gamma\,\beta)$. If we extrapolate this vector from the decay point $(0, 0, l)$, the distance of the closest approach to the origin (the impact parameter b) is given by minimizing

$$b(t)^2 = |(-t\sin\theta, 0, l - t\,\gamma(\cos\theta+\beta))|^2 = t^2\sin^2\theta + l^2 - 2\,l\,t\,\gamma(\cos\theta+\beta) + t^2\,\gamma^2(\cos\theta+\beta)^2$$

$$= t^2\left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right) - 2\,l\,t\,\gamma(\cos\theta+\beta) + l^2$$

The minimum is

$$b^2 = l^2 - (l\,\gamma(\cos\theta+\beta))^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right)$$

$$= l^2\left[1 - \gamma^2(\cos\theta+\beta)^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right)\right]$$

The normalized distribution in the decay length is

$$\frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,l\, e^{-l/\gamma\,c\,\beta\,\tau}$$

In addition, we assume the isotropic decay

$$\frac{1}{2}\int_{-1}^1 d\cos\theta$$

Therefore, in order to find the distribution in the impact parameter, we need to work out

$$\frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,l\, e^{-l/\gamma\,c\,\beta\,\tau}\,\frac{1}{2}\int_{-1}^1 d\cos\theta$$

Now we insert unity and use the expression,

$$\frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,l\, e^{-l/\gamma\,c\,\beta\,\tau}\,\frac{1}{2}\int_{-1}^1 d\cos\theta\int_0^\infty d\,b^2\,\delta\left(b^2 - l^2\left[1 - \gamma^2(\cos\theta+\beta)^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right)\right]\right)$$

We can integrate over l easily because its solution for the delta function is

$$l^2 = b^2 \Big/ \left[1 - \gamma^2(\cos\theta+\beta)^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right)\right].$$

The distribution is then given by

$$\frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,l^2\,\frac{1}{2\,l}\,e^{-l/\gamma\,c\,\beta\,\tau}\,\frac{1}{2}\int_{-1}^1 d\cos\theta\int_0^\infty d\,b^2\,\delta\left(l^2 - b^2 \Big/ \left[1 - \gamma^2(\cos\theta+\beta)^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+\beta)^2\right)\right]\right)\frac{1}{\left[1-\gamma^2(\cos\theta+\beta)^2 \Big/ \left(\sin^2\theta+\gamma^2(\cos\theta+\beta)^2\right)\right]}$$

$$= \frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,b\,b\int_{-1}^1 d\cos\theta\,\frac{l^2}{b^2}\,\frac{1}{2\,l}\,e^{-l/\gamma\,c\,\beta\,\tau}$$

$$= \frac{1}{\gamma\,c\,\beta\,\tau}\int_0^\infty d\,b\int_{-1}^1 d\cos\theta\,\frac{l}{2\,b}\,e^{-l/\gamma\,c\,\beta\,\tau}$$

where l is given by the above solution.

Let us consider the ultrarelativistic limit $\beta \to 1$. In this case, the solution for l simplifies to

$$l^2 = b^2 \Big/ \left[1 - \gamma^2(\cos\theta+1)^2 \Big/ \left(\sin^2\theta + \gamma^2(\cos\theta+1)^2\right)\right].$$

$$= b^2\left(\sin^2\theta + \gamma^2(\cos\theta+1)^2\right) \Big/ \sin^2\theta$$

In addition, we can ignore $\sin^2\theta$ in the numerator for $\gamma \gg 1$, and

$l = b\,\gamma\,(1+\cos\theta)/\sin\theta$.

Then the normalized distribution simplifies drastically to

$$\frac{1}{\gamma c \tau}\int_0^\infty d\,b \int_{-1}^1 d\cos\theta\,\frac{\gamma\,(1+\cos\theta)}{2\sin\theta}\,e^{-b\,(1+\cos\theta)/c\,\tau\sin\theta}$$

$$=\frac{1}{2\,c\,\tau}\int_0^\infty d\,b \int_{-1}^1 d\cos\theta\,\frac{1+\cos\theta}{\sin\theta}\,e^{-b\,(1+\cos\theta)/c\,\tau\sin\theta}$$

$$=\frac{1}{2\,c\,\tau}\int_0^\infty d\,b \int_{-1}^1 d\cos\theta\,\frac{1+\cos\theta}{\sin\theta}\,e^{-b\,(1+\cos\theta)/c\,\tau\sin\theta}$$

$$=\frac{1}{2\,c\,\tau}\int_0^\infty d\,b \int_{-1}^1 d\cos\theta\cot\frac{\theta}{2}\,e^{-b\,\cot(\theta/2)/c\,\tau}$$

Using $t=\cot\frac{\theta}{2}=\frac{1+\cos\theta}{\sin\theta}$,

$$\frac{1}{2\,c\,\tau}\int_0^\infty d\,b \int_0^\infty d\,t\,\frac{4\,t}{(1+t^2)^2}\,t\,e^{-b\,t/c\,\tau}$$

```
Integrate[2/(c τ) t^2/(1 + t^2)^2 E^(-b t/(c τ)), {t, 0, ∞}, Assumptions -> Re[b/(c τ)] > 0]
```

$$\frac{\text{MeijerG}\left[\left\{\left\{-\frac{1}{2}\right\},\{\}\right\},\left\{\left\{0,\frac{1}{2},\frac{1}{2}\right\},\{\}\right\},\frac{b^2}{4\,c^2\,\tau^2}\right]}{c\,\sqrt{\pi}\,\tau}$$

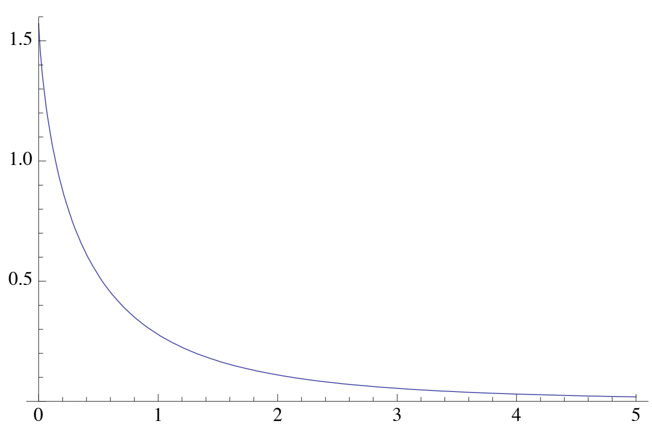
```
Integrate[MeijerG[{{-1/2}, {}}, {{0, 1/2, 1/2}, {}}, b^2/(4 c^2 τ^2)]/(c Sqrt[π] τ), {b, 0, ∞}]
```

$$c\,\sqrt{\frac{1}{c^2\,\tau^2}}\,\tau$$

```
N[MeijerG[{{-1/2}, {}}, {{0, 1/2, 1/2}, {}}, b^2/(4 c^2 τ^2)]/(c Sqrt[π] τ) /. {τ → 1/c} /. {b → 0}]
```

1.5708

```
plot∞ =
 Plot[MeijerG[{{-1/2}, {}}, {{0, 1/2, 1/2}, {}}, b^2/(4 c^2 τ^2)]/(c Sqrt[π] τ) /. {τ → 1/c}, {b, 0, 5}, PlotRange → {0, 1.6}]
```



The result is clearly γ-independent. This is because the large γ would lead to a larger decay length as γ, whie the decay products are more focused as $1/\gamma$ and hence the impact parameter distribution is approximately γ-independent.

For modest boots, we need to numerically integrate over $\cos\theta$.

```
Together[1 - (Cos[θ] + β)^2 / (Sin[θ]^2 + (Cos[θ] + β)^2)]
```

$$\frac{\text{Sin}[\theta]^2}{\beta^2 + 2\,\beta\,\text{Cos}[\theta] + \text{Cos}[\theta]^2 + \text{Sin}[\theta]^2}$$

```
Simplify[%]
```

$$\frac{\text{Sin}[\theta]^2}{1 + \beta^2 + 2\,\beta\,\text{Cos}[\theta]}$$

```
NIntegrate[
  Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2 / (Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /. {β → βγ / Sqrt[1 + βγ^2]} /.
    {γ → Sqrt[1 + βγ^2]} /. {βγ → 100} /. {c → 1, τ → 1} /. {b → 0.001}, {θ, 0, π}]
```

1.55721

```
Solve[β γ == βγ /. {β → Sqrt[1 - γ^-2]}, γ]
```

$$\left\{\left\{\gamma \to -\sqrt{1 + \beta\gamma^2}\right\}, \left\{\gamma \to \sqrt{1 + \beta\gamma^2}\right\}\right\}$$

$$\left\{\left\{\gamma \to -\sqrt{1 + \beta\gamma^2}\right\}, \left\{\gamma \to \sqrt{1 + \beta\gamma^2}\right\}\right\}$$

$$\left\{\left\{\gamma \to -\sqrt{1 + \beta\gamma^2}\right\}, \left\{\gamma \to \sqrt{1 + \beta\gamma^2}\right\}\right\}$$

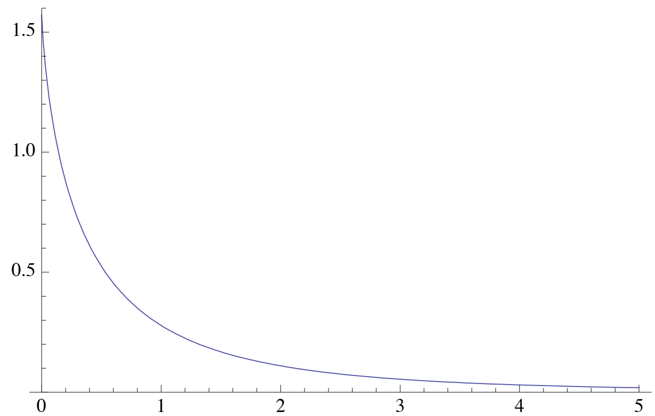
```
β /. {β → Sqrt[1 - γ^-2]} /. %[[2]]
```

$$\sqrt{1 - \frac{1}{1 + \beta\gamma^2}}$$

```
Simplify[%]
```

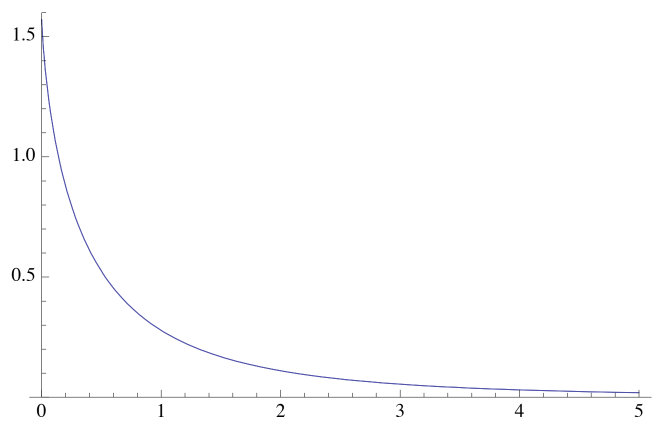
$$\sqrt{\frac{\beta\gamma^2}{1 + \beta\gamma^2}}$$

```
plot1000 = Plot[NIntegrate[
   Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2/(Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /. {β → βγ/Sqrt[1 + βγ^2]} /.
    {γ → Sqrt[1 + βγ^2]} /. {βγ → 1000} /. {c → 1, τ → 1}, {θ, 0, π}], {b, 0, 5}, PlotRange → {0, 1.6}]
```



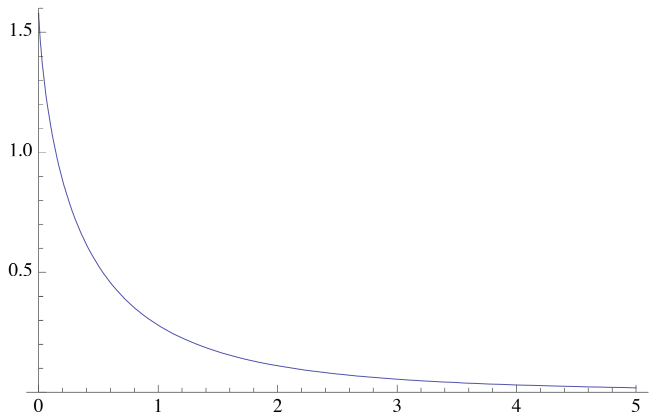
Just to verify that the numerical integral for this large value of γ is consistent with the analytic result for large γ:

```
Show[plot∞, plot1000]
```

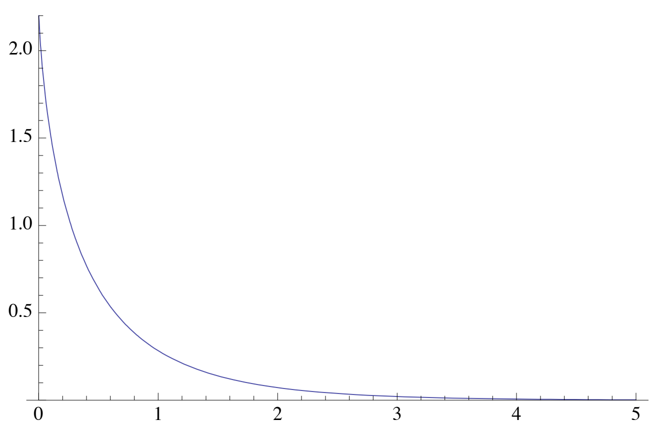


They are indistinguishable.

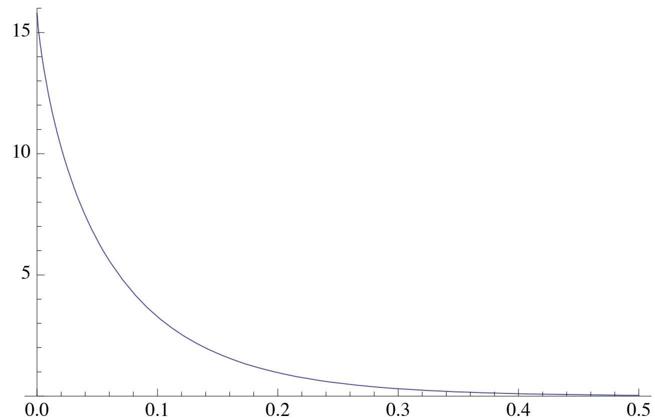
```
plot10 = Plot[NIntegrate[
   Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2/(Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /. {β → βγ/Sqrt[1 + βγ^2]} /.
     {γ → Sqrt[1 + βγ^2]} /. {βγ → 10} /. {c → 1, τ → 1}, {θ, 0, π}], {b, 0, 5}, PlotRange → {0, 1.6}]
```



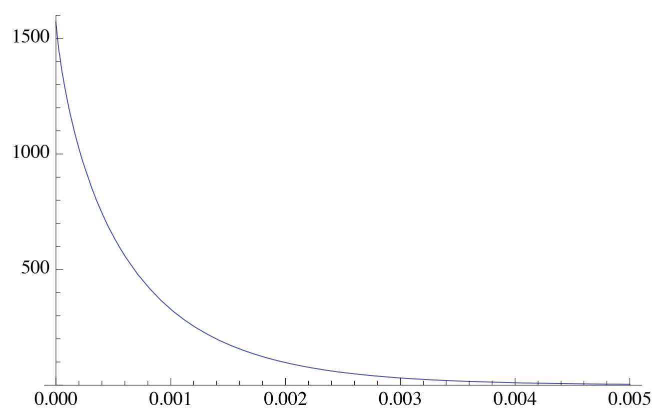
```
plot1 = Plot[NIntegrate[
   Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2/(Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /. {β → βγ/Sqrt[1 + βγ^2]} /.
     {γ → Sqrt[1 + βγ^2]} /. {βγ → 1} /. {c → 1, τ → 1}, {θ, 0, π}], {b, 0, 5}, PlotRange → {0, 2.2}]
```



```
plot01 = Plot[NIntegrate[
   Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2/(Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /. {β → βγ/Sqrt[1 + βγ^2]} /.
     {γ → Sqrt[1 + βγ^2]} /. {βγ → 0.1} /. {c → 1, τ → 1}, {θ, 0, π}], {b, 0, 0.5}, PlotRange → {0, 16}]
```



```
plot001 =
 Plot[NIntegrate[Sin[θ] 1/(γ c β τ) L/(2 b) E^(-L/(γ c β τ)) /. {L → b / Sqrt[1 - γ^2 (Cos[θ] + β)^2/(Sin[θ]^2 + γ^2 (Cos[θ] + β)^2)]} /.
       {β → βγ/Sqrt[1 + βγ^2]} /. {γ → Sqrt[1 + βγ^2]} /. {βγ → 0.001} /.
    {c → 1, τ → 1}, {θ, 0, π}], {b, 0, 5 10^-3}, PlotRange → {0, 1600}]
```



(3) $h \to W^+ W^-$ decay