Notes on Phase Space

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1 Two-Body Phase Space

The two-body phase is the basis of computing higher body phase spaces. We compute it in the rest frame of the two-body system, $P = p_1 + p_2 = (\sqrt{s}, 0, 0, 0)$.

$$
\begin{aligned}
\int d\Phi_2(p_1, p_2) &= \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(P^\mu - p_1^\mu - p_2^\mu) \\
&= \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(\sqrt{s} - E_1 - E_2) \delta(\vec{p}_1 + \vec{p}_2) \\
&= \int \frac{d^3p}{(2\pi)^3 2\sqrt{m_1^2 + \vec{p}^2}} \frac{1}{2\sqrt{m_2^2 + \vec{p}^2}} (2\pi)\delta\left(\sqrt{s} - \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2}\right). \\
&= \int \frac{p^2 dp \; d\cos\theta \; d\phi}{(2\pi)^3 2\sqrt{m_1^2 + p^2}} \frac{1}{2\sqrt{m_2^2 + p^2}} (2\pi)\delta\left(\sqrt{s} - \sqrt{m_1^2 + p^2} - \sqrt{m_2^2 + p^2}\right).
\end{aligned}
\tag{1}
$$

It takes some work to solve the delta function. The end result is that

$$
p = \frac{\sqrt{s}}{2} \bar{\beta} \tag{2}
$$

with

$$
\bar{\beta} = \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{s} + \frac{(m_1^2 - m_2^2)^2}{s^2}} \; . \tag{3}
$$

Then the delta function becomes

$$
\delta\left(\sqrt{s} - \sqrt{m_1^2 + p^2} - \sqrt{m_2^2 + p^2}\right) = \frac{\delta(p - \bar{\beta}\sqrt{s}/2)}{(p/E_1) + (p/E_2)} \; . \tag{4}
$$

Therefore,

$$
\int d\Phi_2(p_1, p_2) \;\; = \;\; \int \frac{d\cos\theta \; d\phi}{(2\pi)^3} \; \frac{p^2}{4E_1 E_2} 2\pi \frac{1}{(p/E_1) + (p/E_2)} \Bigg|_{p = \bar{\beta}\sqrt{s}/2, E_i = \sqrt{m_i^2 + p^2}}
$$

$$
\begin{aligned}
&= \int \frac{d\cos\theta\ d\phi}{(2\pi)^2}\ \frac{p}{4(E_1+E_2)}\bigg|_{p=\bar{\beta}\sqrt{s}/2, E_i=\sqrt{m_i^2+p^2}} \\
&= \frac{\bar{\beta}}{8\pi}\int \frac{d\cos\theta}{2}\frac{d\phi}{2\pi}\ . \qquad (5)
\end{aligned}
$$

It is also useful to remember

$$
E_1 = \frac{\sqrt{s}}{2}\left(1+\frac{m_1^2}{s}-\frac{m_2^2}{s}\right), \qquad (6)
$$

$$
E_2 = \frac{\sqrt{s}}{2}\left(1+\frac{m_2^2}{s}-\frac{m_1^2}{s}\right). \qquad (7)
$$

It is worthwhile looking at two special cases. One is when two masses are equal $m_1 = m_2 = m$. Then

$$
\bar{\beta} = \sqrt{1-\frac{4m^2}{s}} = \sqrt{1-\frac{m^2}{E^2}}\ . \qquad (8)
$$

This is nothing but $\beta = v/c$; hence the notation. However, for $m_1 \neq m_2$, two particles have different velocities and $\bar{\beta}$ cannot be interpreted as the velocity either. The other is when one of the masses vanishes, $m_2 = 0$. Then

$$
\bar{\beta} = 1 - \frac{m_1^2}{s}\ . \qquad (9)
$$

This is nothing but $2E_2/\sqrt{s}$ because $E_2 = |\vec{p}_2| = |\vec{p}_1|$.

2 Decomposing Phase Space into Two-Body Ones

It is often useful to decompose multi-body phase space integral into a product of two-body phase space integrals. As an example, let us decompose a four-body phase space into a produce of three two-body phase spaces. This is very useful if one considers a production of two particles, each of which subsequently decays into two-body state.

The full four-body phase space is given by

$$
\int d\Phi_4 = \int \prod_{i=1}^{4} \frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta(P-p_1-p_2-p_3-p_4). \qquad (10)
$$

We first insert two four-momentum integrals with $q_{12} = p_1 + p_2$ and $q_{34} = p_3 + p_4$,

$$1 = \int \frac{d^4 q_{12}}{(2\pi)^4} \frac{d^4 q_{34}}{(2\pi)^4} (2\pi)^4 \delta(q_{12} - p_1 - p_2) \theta(q_{12}^0) (2\pi)^4 \delta(q_{342} - p_3 - p_4) \theta(q_{34}^0). \quad (11)$$

The step function $\theta(x)$ is 1 for $x > 0$ and 0 for $x < 0$. Even though the step function eliminates a half of the integral over q_{12}^0 and q_{34}^0, the delta function ensures that they are given by the sum of two energies and hence its support is in the half that is retained. In addition, we also insert

$$1 = \int \frac{ds_{12}}{2\pi} \frac{ds_{34}}{2\pi} 2\pi \delta(s_{12} - q_{12}^2) 2\pi \delta(s_{34} - q_{34}^2). \quad (12)$$

With the delta functions, we can regard s_{12} as "mass squared" of the "particle" whose four-momentum is q_{34}^μ. Then one can perform the energy integral and find

$$\begin{aligned} &\int \frac{d^4 q_{12}}{(2\pi)^4} \frac{ds_{12}}{2\pi} (2\pi)^4 \delta(q_{12} - p_1 - p_2) \theta(q_{12}^0) 2\pi \delta(s_{12} - q_{12}^2) \\ &\quad = \int \frac{ds_{12}}{2\pi} \frac{d^3 q_{12}}{(2\pi)^3 2E_{12}} (2\pi)^4 \delta(q_{12} - p_1 - p_2). \end{aligned} \quad (13)$$

Here, we used the delta function $s_{12} - q_{12}^2 = (s_{12} + \vec{q}_{12}^{\,2}) - E_{12}^2 = 0$ and eliminated $E_{12} = q_{12}^0$ with the condition by the step function $q_{12}^0 > 0$. In the last expression, it is understod that the four-dimensional delta function contains $q_{12}^0 = +\sqrt{s_{12} + \vec{q}_{12}^{\,2}}$. We do the same with q_{34} as well. Putting all of the above together, we find

$$\begin{aligned} \int d\Phi_4 = &\int \prod_{i=1}^{4} \frac{d^3 p_i}{(2\pi)^3 2E_i} \frac{ds_{12}}{2\pi} \frac{d^3 q_{12}}{(2\pi)^3 2E_{12}} \frac{ds_{34}}{2\pi} \frac{d^3 q_{34}}{(2\pi)^3 2E_{34}} \\ &(2\pi)^4 \delta(q_{12} - p_1 - p_2) (2\pi)^4 \delta(q_{34} - p_3 - p_4) (2\pi)^4 \delta(P - q_{12} - q_{34}). \end{aligned} \quad (14)$$

Now, note that the integration volumes $d^3 p_i / (2\pi)^3 2E_i$ is Lorentz invariant. Therefore, we can carry out these integrals in any frame we wish. In particular, we take the "rest frame" of q_{12} to perform p_1 and p_2 integrals. In this frame, we denote the momenta as $\hat{p}_{1,2}$, and we know that

$$\begin{aligned} \int d\Phi_2(\hat{p}_1, \hat{p}_2) &= \int \frac{d^3 \hat{p}_1}{(2\pi)^3 2\hat{E}_1} \frac{d^3 \hat{p}_2}{(2\pi)^3 2\hat{E}_2} (2\pi)^4 \delta(\hat{E_{12}} - \hat{E}_1 - \hat{E}_2) \delta(\hat{\vec{p}}_1 + \hat{\vec{p}}_2) \\ &= \frac{\bar{\beta}_{12}}{8\pi} \int \frac{d\cos\hat{\theta}_{12}}{2} \frac{d\hat{\phi}_{12}}{2\pi}, \end{aligned} \quad (15)$$

with

$$\bar{\beta}_{12} = \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{s_{12}} + \frac{(m_1^2 - m_2^2)^2}{s_{12}^2}} \ . \tag{16}$$

The same is true with the p_3 and p_4 integrals, where the "rest frame" of q_{34} can be chosen. (This *different* reference frame is also denoted with the same hatted variables. Any hatted variables that refer to 1 and 2 are in the rest frame of q_{12}, while those that refer to 3 and 4 are in the rest frame of q_{34}.) Therefore, the whole four-body phase space is now reduced to

$$\begin{aligned} \int d\Phi_4 = \int \frac{ds_{12}}{2\pi} \frac{d^3 q_{12}}{(2\pi)^3 2E_{12}} \frac{ds_{34}}{2\pi} \frac{d^3 q_{34}}{(2\pi)^3 2E_{34}} (2\pi)^4 \delta(P - q_{12} - q_{34}) \\ \frac{\bar{\beta}_{12}}{8\pi} \int \frac{d\cos\hat{\theta}_{12}}{2} \frac{d\hat{\phi}_{12}}{2\pi} \frac{\bar{\beta}_{34}}{8\pi} \int \frac{d\cos\hat{\theta}_{34}}{2} \frac{d\hat{\phi}_{34}}{2\pi} \ . \end{aligned} \tag{17}$$

On the other hand, the integrals over q_{12} and q_{34} can be done as if each of them is a "particle" of mass $\sqrt{s_{12}}$ and $\sqrt{s_{34}}$, and are done in the rest frame of P (which is the lab frame for a symmetric collider and we do not put a hat on the variables in this frame). Therefore,

$$\begin{aligned} \int d\Phi_2(q_{12}, q_{34}) &= \int \frac{d^3 q_{12}}{(2\pi)^3 2E_{12}} \frac{d^3 q_{34}}{(2\pi)^3 2E_{34}} (2\pi)^4 \delta(P - q_{12} - q_{34}) \\ &= \frac{\bar{\beta}}{8\pi} \int \frac{d\cos\theta}{2} \frac{d\phi}{2\pi} \ , \end{aligned} \tag{18}$$

where

$$\bar{\beta} = \sqrt{1 - \frac{2(s_{12} + s_{34})}{s} + \frac{(s_{12} - s_{34})^2}{s^2}} \ . \tag{19}$$

Putting everything together, we find

$$\begin{aligned} \int d\Phi_4 &= \int \frac{ds_{12}}{2\pi} \frac{ds_{34}}{2\pi} \frac{\bar{\beta}}{8\pi} \int \frac{d\cos\theta}{2} \frac{d\phi}{2\pi} \frac{\bar{\beta}_{12}}{8\pi} \int \frac{d\cos\hat{\theta}_{12}}{2} \frac{d\hat{\phi}_{12}}{2\pi} \frac{\bar{\beta}_{34}}{8\pi} \int \frac{d\cos\hat{\theta}_{34}}{2} \frac{d\hat{\phi}_{34}}{2\pi} \\ &= \int \frac{ds_{12}}{2\pi} \frac{ds_{34}}{2\pi} d\Phi_2(q_{12}, q_{34}) d\Phi_2(\hat{p}_1, \hat{p}_2) d\Phi_2(\hat{p}_3, \hat{p}_4). \end{aligned} \tag{20}$$

3 Narrow-Width Approximation

The narrow-width approximation uses the fact that the amplitude through a pole of unstable particle decouples as

$$\mathcal{M} = \sum_{h_a, h_b} \frac{\mathcal{M}(i \to a + b) \mathcal{M}(a \to 1 + 2) \mathcal{M}(b \to 3 + 4)}{(s_{12} - m_a^2 + i m_a \Gamma_a)(s_{34} - m_b^2 + i m_b \Gamma_b)} \ . \tag{21}$$

For an unstable scalar, it is obvious. For an unstable fermion, we use the fact that $\not{p} + m = \sum_{h=\pm 1/2} u_h(p)\bar{u}_h(p)$ if the four-momentum is on-shell $p^2 = m^2$ and has positive energy $p^0 > 0$. If it is one-shell with the negative energy $p^0 < 0$, $\not{p} + m = -\sum_{h=\pm 1/2} v_h(-p)\bar{v}_h(-p)$. Of course, the four-momentum is not exactly one-shell, but we ignore the off-shellness in the numerator. Then one of the spinors is regarded as a part of the production amplitude, the other of the decay amplitude. The same idea applies to vector bosons, where we use $-g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2} = \sum_{h=\pm 1,0} \epsilon_h^\mu(q)\epsilon_h^{\nu *}(q)$.

When we integrate the squared amplitude on the four-body phase space, we use the two-body decomposition (so far not summed over initial and final state helicities)

$$
\begin{aligned}
\sigma &= \frac{1}{2s\bar{\beta}_i}\int d\Phi_4 |\mathcal{M}|^2 \\
&= \frac{1}{2s\bar{\beta}_i}\int \frac{ds_{12}}{2\pi}\frac{ds_{34}}{2\pi} d\Phi_2(q_{12}, q_{34}) d\Phi_2(\hat{p}_1, \hat{p}_2) d\Phi_2(\hat{p}_3, \hat{p}_4) \\
&\quad \left| \sum_{h_a, h_b} \frac{\mathcal{M}(i \to a+b)\mathcal{M}(a \to 1+2)\mathcal{M}(b \to 3+4)}{(s_{12} - m_a^2 + i m_a\Gamma_a)(s_{34} - m_b^2 + i m_b \Gamma_b)} \right|^2 \\
&= \frac{1}{2s\bar{\beta}_i}\int \frac{ds_{12}}{2\pi}\frac{ds_{34}}{2\pi} d\Phi_2(q_{12}, q_{34}) d\Phi_2(\hat{p}_1, \hat{p}_2) d\Phi_2(\hat{p}_3, \hat{p}_4) \\
&\quad \frac{\left|\sum_{h_a, h_b} \mathcal{M}(i \to a+b)\mathcal{M}(a \to 1+2)\mathcal{M}(b \to 3+4)\right|^2}{[(s_{12} - m_a^2)^2 + m_a^2\Gamma_a^2][(s_{34} - m_b^2)^2 + m_b^2\Gamma_b^2]} \\
&= \frac{1}{2s\bar{\beta}_i}\int d\Phi_2(q_{12}, q_{34}) d\Phi_2(\hat{p}_1, \hat{p}_2) d\Phi_2(\hat{p}_3, \hat{p}_4) \\
&\quad \frac{1}{2m_a\Gamma_a}\frac{1}{2m_b\Gamma_b} \left| \sum_{h_a, h_b} \mathcal{M}(i \to a+b)\mathcal{M}(a \to 1+2)\mathcal{M}(b \to 3+4) \right|^2
\end{aligned}
\tag{22}
$$

In the last step, we assumed that the amplitudes depend very little on s_{12}, s_{34} within their widths. When the detailed distributions are not of interest, the interference term among different helicities of the decaying particle oscillates over the phase space and drops out. Therefore, the total cross section reduces to

$$
\sigma = \frac{1}{2s\bar{\beta}_i}\int d\Phi_2(q_{12}, q_{34}) d\Phi_2(\hat{p}_1, \hat{p}_2) d\Phi_2(\hat{p}_3, \hat{p}_4)
$$

$$
\begin{aligned}
&\frac{1}{2m_a\Gamma_a}\frac{1}{2m_b\Gamma_b}\sum_{h_a,h_b}|\mathcal{M}(i\to a+b)\mathcal{M}(a\to 1+2)\mathcal{M}(b\to 3+4)|^2 \\
= &\frac{1}{2s\bar{\beta}_i}\int d\Phi_2(q_{12},q_{34})\sum_{h_a,h_b}|\mathcal{M}(i\to a+b)|^2 B(a\to 1+2)B(b\to 3+4).
\end{aligned}
\tag{23}
$$

In the last step, we used the definition of the partial width

$$
\Gamma(a\to 1+2)=\frac{1}{2m_a}\int d\Phi_2(\hat{p}_1,\hat{p}_2)\,|\mathcal{M}(a\to 1+2)|^2 \tag{24}
$$

and the branching fraction

$$
B(a\to 1+2)=\frac{\Gamma(a\to 1+2)}{\Gamma_a}\ . \tag{25}
$$

4 Three-Body Phase Space

Three-body decays are often important. Examples include $\mu\to e\bar{\nu}_e\nu_\mu$, $\omega\to 3\pi$, $H\to WW^*$. In these examples, no two-particle combination hits a pole of an unstable particle. Nonetheless the phase space can be worked out the same way we did earlier.

Using the two-body phase space decomposition just once, we find

$$
\begin{aligned}
\int d\Phi_3 &= \int\frac{ds_{23}}{2\pi}d\Phi_2(p_1,q_{23})d\Phi_2(p_2,p_3) \\
&= \int\frac{ds_{23}}{2\pi}\frac{d\cos\theta_1}{2}\frac{d\phi_1}{2\pi}\frac{\bar{\beta}_1(\frac{m_1^2}{s},\frac{s_{23}}{s})}{8\pi}\frac{d\cos\hat{\theta}_{23}}{2}\frac{d\hat{\phi}_{23}}{2\pi}\frac{\bar{\beta}_{23}(\frac{m_2^2}{s_{23}},\frac{m_3^2}{s_{23}})}{8\pi}\ .
\end{aligned}
\tag{26}
$$

Often we are interested in a decay of unpolarized particles (spin averaged), and the distribution is isotropic. Then the overall rotation of the system $(\cos\theta_1,\phi_1)$ integrals can be dropped. It is convenient to define the polar angle $\hat{\theta}_{23}$ relative to the direction $-\vec{p}_1$. Then the azimuthal dependence on $\hat{\phi}_{23}$ is also trivial as it corresponds to the overall rotation of the system around the axis $-\vec{p}_1$.

The variables s_{23} and $\cos\hat{\theta}_{23}$ are often rewritten with the energy fractions $x_{1,2,3}$. (Or, more traditional variables are $m_{12}^2=s_{12}$ and $m_{23}^2=s_{23}$ called Dalitz variables.) First,

$$
x_1=\frac{E_1}{\sqrt{s}/2}=1+\frac{m_1^2}{s}-\frac{s_{23}}{s}\ . \tag{27}
$$

Second, using the assumed isotropy, we can make q_{23} point along the positive z-direction,

$$p_1 = \frac{\sqrt{s}}{2}(1 + \frac{m_1^2}{s} - \frac{s_{23}}{s}, 0, 0, -\bar{\beta}_1), \tag{28}$$

$$q_{23} = \frac{\sqrt{s}}{2}(1 - \frac{m_1^2}{s} + \frac{s_{23}}{s}, 0, 0, \bar{\beta}_1). \tag{29}$$

Therefore, the boost factor from the rest frame of q_{23} to the center-of-momentum frame is given by

$$\gamma = \frac{E_{23}}{\sqrt{s_{23}}} = \frac{\sqrt{s}}{2\sqrt{s_{23}}}\left(1 - \frac{m_1^2}{s} + \frac{s_{23}}{s}\right), \tag{30}$$

$$\gamma\beta = \frac{\sqrt{s}}{2\sqrt{s_{23}}}\bar{\beta}_1 . \tag{31}$$

The four-momentum of the particle 2 in the rest frame of q_{23} is

$$\hat{p}_2 = \frac{\sqrt{s_{23}}}{2}\left(1 + \frac{m_2^2}{s_{23}} - \frac{m_3^2}{s_{23}}, \bar{\beta}_{23}\sin\hat{\theta}_{23}, 0, \bar{\beta}_{23}\cos\hat{\theta}_{23}\right). \tag{32}$$

The energy of the particle 2 in the center-of-momentum frame is then

$$E_2 = \frac{\sqrt{s_{23}}}{2}\left[\gamma\left(1 + \frac{m_2^2}{s_{23}} - \frac{m_3^2}{s_{23}}\right) + \gamma\beta\bar{\beta}_{23}\cos\hat{\theta}_{23}\right], \tag{33}$$

and hence

$$x_2 = \frac{\sqrt{s_{23}}}{\sqrt{s}}\left[\gamma\left(1 + \frac{m_2^2}{s_{23}} - \frac{m_3^2}{s_{23}}\right) + \gamma\beta\bar{\beta}_{23}\cos\hat{\theta}_{23}\right]. \tag{34}$$

Let us specifically study the case of massless particles $m_{1,2,3} = 0$. Performing all the integrals for overall rotations, we have

$$\int d\Phi_3 = \int \frac{ds_{23}}{2\pi}\frac{1}{8\pi}\left(1 - \frac{s_{23}}{s}\right)\frac{d\cos\hat{\theta}_{23}}{2}\frac{1}{8\pi} . \tag{35}$$

The energies of the particles 1 and 2 are

$$x_1 = 1 - \frac{s_{23}}{s} , \tag{36}$$

$$\begin{aligned} x_2 &= \frac{\sqrt{s_{23}}}{\sqrt{s}}\left[\frac{\sqrt{s}}{2\sqrt{s_{23}}}\left(1 + \frac{s_{23}}{s}\right) + \frac{\sqrt{s}}{2\sqrt{s_{23}}}\left(1 - \frac{s_{23}}{s}\right)\cos\hat{\theta}_{23}\right] \\ &= \frac{1}{2}\left[\left(1 + \frac{s_{23}}{s}\right) + \left(1 - \frac{s_{23}}{s}\right)\cos\hat{\theta}_{23}\right] = \frac{1}{2}(2 - x_1 + x_1\cos\hat{\theta}_{23}). \end{aligned} \tag{37}$$

Therefore, the three-body phase space becomes simply

$$\int d\Phi_3 = \frac{s}{128\pi^3} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \ . \tag{38}$$

Obviously, $x_1 + x_2 + x_3 = 2$ and the phase space is a triangle on this plane in the three-dimensional space.

Note that the matrix element $|\mathcal{M}|^2$ in general depends on x_i in a non-trivial way. The expression there is only the phase space. For instance, in the decay $H \to WW^* \to W\ell\nu_\ell$, the virtual W propagator prefers $m_{\ell\nu_\ell}$ to be as close as possible to m_W, giving a non-trivial x_W dependence.

With finite masses, the edges and vertices of the triangle get rounded. For details, there is a PDG review article.

5 Decaying Particles in the Lab Frame

It is important to understand the kinematics of the decay products of a particle of mass M decaying in flight.

For a isotropic two-body decay, we start with the simple two-body phase space. To bring it to the lab frame, we boost the four-momentum vector from the rest frame

$$p_1 = \frac{M}{2}(1 + \frac{m_1^2}{M^2} - \frac{m_2^2}{M^2}, \bar{\beta}\sin\hat{\theta}, 0, \bar{\beta}\cos\hat{\theta}). \tag{39}$$

Here, we used our liberty to choose the origin of the azimuth such that the y-component of the vector vanishes. To go to the lab frame, we boost it with

$$\gamma = \frac{E}{M}, \qquad \beta = \sqrt{1-\gamma^{-2}} \ . \tag{40}$$

The energy of the particle 1 in the lab frame is then

$$E_1 = \frac{M}{2}\left[\gamma\left(1 + \frac{m_1^2}{M^2} - \frac{m_2^2}{M^2}\right) + \gamma\beta\bar{\beta}\cos\hat{\theta}\right]. \tag{41}$$

The important point here is the linear relationship between E_1 and $\cos\hat{\theta}$. In particular, for the isotropic decay, the distribution is flat in $-1 \leq \cos\hat{\theta} \leq 1$, and therefore we obtain a flat distribution in E_1 for the range

$$\left(1 + \frac{m_1^2}{M^2} - \frac{m_2^2}{M^2}\right) - \beta\bar{\beta} \leq \frac{2E_1}{E} \leq \left(1 + \frac{m_1^2}{M^2} - \frac{m_2^2}{M^2}\right) + \beta\bar{\beta}. \tag{42}$$

In particular for the massless case $m_1 = m_2 = 0$, it is particularly simple,

$$1 - \beta \leq \frac{2E_1}{E} \leq 1 + \beta. \tag{43}$$

It is useful to define the energy fraction $x_1 = E_1/E$, and we find a flat distribution in $\frac{1-\beta}{2} \leq x_1 \leq \frac{1+\beta}{2}$.

For an isotropic three-body decay into massless particles, we focus on the particle 1 (say electron in the decay $\tau^- \to e^- \bar{\nu}_e \nu_\tau$) whose phase space is

$$\int d\Phi_3 = \frac{M^2}{128\pi^3} \int_0^1 d\hat{x}_1 \hat{x}_1 \int_{-1}^{1} \frac{d\cos\hat{\theta}}{2} \ . \tag{44}$$

The four-momentum of the particle 1 in the rest frame is

$$\hat{p}_1 = \frac{M}{2} \hat{x}_1 (1, \sin\hat{\theta}, 0, \cos\hat{\theta}). \tag{45}$$

In the lab frame, its energy is

$$E_1 = E x_1 = \frac{M}{2} \gamma \hat{x}_1 (1 + \beta \cos\hat{\theta}). \tag{46}$$

For simplicity, we consider the case $\beta \approx 1$. Then we can stick in a delta function and obtain

$$\begin{aligned}
\int d\Phi_3 &= \frac{M^2}{128\pi^3} \int_0^1 d\hat{x}_1 \hat{x}_1 \int_{-1}^{1} \frac{d\cos\hat{\theta}}{2} \int dE_1 \delta(\frac{M}{2}\gamma\hat{x}_1(1+\cos\hat{\theta}) - E_1) \\
&= \frac{M^2}{128\pi^3} \int dE_1 \int_0^1 d\hat{x}_1 \hat{x}_1 \int_{-1}^{1} \frac{d\cos\hat{\theta}}{2} \frac{\delta(\cos\hat{\theta} + 1 - 2E_1/(M\gamma\hat{x}_1))}{M\gamma\hat{x}_1/2} \\
&= \frac{M}{128\pi^3\gamma} \int dE_1 \int_{E_1/M\gamma}^{1} d\hat{x}_1 \\
&= \frac{M}{128\pi^3\gamma} \int dE_1 \left(1 - \frac{E_1}{M\gamma}\right) \\
&= \frac{M^2}{128\pi^3} \int dx_1 (1 - x_1). \qquad (47)
\end{aligned}$$

This is a simple downward triangular distribution.